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Geometric conditions for G^3 continuity of surfaces

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Abstract

This paper gives (necessary and sufficient) geometric conditions for G^3 continuity of surfaces. These conditions are based on the G^2 continuity of characteristic surface curves, namely lines of curvature. Furthermore, the curvature of the lines of curvature is used for visualizing continuity properties of surfaces. © 1997 Elsevier Science B.V.

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1. Introduction

Geometric continuity of curves and surfaces is an important and widely researched concept in Computer Aided Geometric Design (CAGD). Several definitions of geometric continuity have been introduced. Here we want to use the following popular definition (see (Farin, 1992; Pottmann, 1988)):

Two curves are G^r at a common point $\boldsymbol{x}$ iff there exists a regular parametrization with respect to which they are C^r at $\boldsymbol{x}$. Two surfaces are G^r along a common line l iff there exists a regular parametrization with respect to which they are C^r along l.

The advantage of this definition is that it works for higher order continuities. On the other hand, it is rather abstract if we really want to check if a curve/surface is G^r. It has been recognized that there are equivalent definitions for $r = 1, 2$ which use geometric properties of the curve/surface:

Two curves through the point $\boldsymbol{x}_0$ are G^2 in this point iff

- the normalized tangent vectors coincide in $\boldsymbol{x}_0$ and
- the osculating planes coincide in $\boldsymbol{x}_0$ and
- the signed curvatures coincide in $\boldsymbol{x}_0$.

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Two surfaces sharing a common line l are G^1 along l iff their normalized normal vectors coincide along l.

Two surfaces are G^2 along l iff

- the normalized normal vectors coincide along l and
- the Dupin's indicatrices coincide along l.

A survey on geometric continuity issues can be found in (Farin, 1992; Gregory, 1989). In (Pottmann, 1988) conditions for G^3 of curves and tensor product surfaces described in Bézier form are shown. In (Pegna and Wolter, 1992) there are more geometric conditions for G^2 surfaces formulated.

In this paper we introduce (necessary and sufficient) geometric conditions for G^3 continuity of surfaces. These conditions are based on G^2 continuity of lines of curvature. The conditions are formulated in Theorem 1. An application of these geometric conditions for G^3 continuity of surfaces is shown in Section 4. Here we use the curvature of the lines of curvature as a surface interrogation method. The visualizations obtained this way show shape and continuity features of the surface and detect umbilical points.

Notation and abbreviations:

Let

$$\boldsymbol{n} = \boldsymbol{n}(u,v) = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}$$

be the *normalized normal vector* of the surface $\boldsymbol{x}(u,v)$. Furthermore, we use the classical abbreviations

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v,$$

$$L = \boldsymbol{n} \cdot \boldsymbol{x}_{uu}, \quad M = \boldsymbol{n} \cdot \boldsymbol{x}_{uv}, \quad N = \boldsymbol{n} \cdot \boldsymbol{x}_{vv}$$

and their partial derivatives.

2. Theoretical background

Since we want to deal with G^2 of lines of curvature and asymptotic lines we have to show how to compute their curvature. It turns out that we can compute their curvatures even if a closed parametric form of these curves does not exist.

Lines of curvature can be considered as *tangent curves of vector fields*. To show this, we start with the treatment of 2D vector fields and their tangent curves:

Given is a 2D vector field $V : \mathbb{E}^2 \to \mathbb{R}^2$. V assigns a vector $(vx(P), vy(P))^{\mathrm{T}}$ to any point $P \sim (u,v)$. We use the notation $V(P) = V(u,v) = (vx(u,v), vy(u,v))^{\mathrm{T}}$. A point $P \in \mathbb{E}^2$ is called *critical point of* V if $V(P) = \mathbf{0}$ is the zero vector. A curve $\boldsymbol{t} \subseteq \mathbb{E}^2$ is called *tangent curve* (stream line, flow line, characteristic curve) of the vector field V if the following condition is satisfied: For all points $P \in \boldsymbol{t}$, the tangent vector of the curve in the point P has the same direction as the vector $V(P)$. For every point $P \in \mathbb{E}^2$ there is one and only one tangent curve through it (except for critical points of V). Tangent curves do not intersect each other (except for critical points of V).

Given a noncritical point $P_0 \sim (u_0, v_0)$ in the vector field, we want to compute the first and second derivative vector of the tangent curve through P_0 in the point P_0. Let $\boldsymbol{t}(t)$ be the tangent curve through P_0 and $\boldsymbol{t}(t_0) = P_0$. From the definition of the tangent curves we know about the first derivative vector of $\boldsymbol{t}$ at $t = t_0$:

$$\dot{\boldsymbol{t}}(t_0) = V(\boldsymbol{t}(t_0)) = V(P_0). \tag{1}$$

Applying the chain rule to $\dot{\boldsymbol{t}}(t) = V(\boldsymbol{t}(t))$, we obtain for the second derivative vector of $\boldsymbol{t}$ at $t = t_0$:

$$\ddot{\boldsymbol{t}}(t_0) = V_u \cdot \frac{\mathrm{d}u}{\mathrm{d}t}(t_0) + V_v \cdot \frac{\mathrm{d}v}{\mathrm{d}t}(t_0) = (vx \cdot V_u + vy \cdot V_v)(P_0). \tag{2}$$

Now we consider a surface $\boldsymbol{x}$ over the (u, v)-domain of the vector field V. This way the tangent curves of V are mapped to surface curves on $\boldsymbol{x}(u, v)$. Let $(u(t), v(t))$ be a curve in the domain of $\boldsymbol{x}$. Then we know about the map of this curve onto the surface and its first and second derivative (see (Farin, 1992)):

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{x}(u(t), v(t)), \\ \dot{\boldsymbol{x}}(t) &= (\dot{u} \cdot \boldsymbol{x}_u + \dot{v} \cdot \boldsymbol{x}_v)(t), \\ \ddot{\boldsymbol{x}}(t) &= (\ddot{u} \cdot \boldsymbol{x}_u + \ddot{v} \cdot \boldsymbol{x}_v + \dot{u}^2 \cdot \boldsymbol{x}_{uu} + 2 \cdot \dot{u} \cdot \dot{v} \cdot \boldsymbol{x}_{uv} + \dot{v}^2 \cdot \boldsymbol{x}_{vv})(t). \end{aligned} \tag{3}$$

Considering the domain curves as the tangent curves of V, we obtain from (1) and (2):

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = V, \quad \begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} = vx \cdot V_u + vy \cdot V_v.$$

Inserting this into (3) gives

$$\dot{\boldsymbol{x}} = vx \cdot \boldsymbol{x}_u + vy \cdot \boldsymbol{x}_v, \tag{4}$$

$$\begin{aligned} \ddot{\boldsymbol{x}} = {} & (vx \cdot vx_u + vy \cdot vx_v) \cdot \boldsymbol{x}_u + (vx \cdot vy_u + vy \cdot vy_v) \cdot \boldsymbol{x}_v \\ & + vx^2 \cdot \boldsymbol{x}_{uu} + 2 \cdot vx \cdot vy \cdot \boldsymbol{x}_{uv} + vy^2 \cdot \boldsymbol{x}_{vv}. \end{aligned} \tag{5}$$

Now we can compute the curvatures of the tangent curves through every surface point:

$$\kappa = \frac{\|\dot{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}\|}{\|\dot{\boldsymbol{x}}\|^3}. \tag{6}$$

(4)–(6) yield the following statement: in order to compute the curvatures of tangent curves on a surface $\boldsymbol{x}$, we only have to know the vector field V (and its partial derivatives) in the domain of $\boldsymbol{x}$ which produces the tangent curves.

3. Lines of curvature

Lines of curvature are the tangent curves of the principal directions—considered as a vector field on the surface. Since there are two principal direction vector fields (whose vectors are perpendicular to each other) we have two families of lines of curvature.

The principal directions in the domain of $\boldsymbol{x}$ are the solutions $(vx, vy)^{\mathrm{T}}$ of the quadratic equation (see (Farin, 1992)):

$$\det \begin{bmatrix} vy^2 & -vx \cdot vy & vx^2 \\ E & F & G \\ L & M & N \end{bmatrix} = 0. \tag{7}$$

(7) yields two solution classes of $(vx, vy)^{\mathrm{T}}$ (where a solution class contains only vectors of the same direction). We use the abbreviations ha, hb and hc which are defined as:

$$\begin{pmatrix} ha \\ hb \\ hc \end{pmatrix} = \begin{pmatrix} E \\ F \\ G \end{pmatrix} \times \begin{pmatrix} L \\ M \\ N \end{pmatrix}. \tag{8}$$

This gives for the partial derivatives:

$$\begin{pmatrix} ha_u \\ hb_u \\ hc_u \end{pmatrix} = \begin{pmatrix} E_u \\ F_u \\ G_u \end{pmatrix} \times \begin{pmatrix} L \\ M \\ N \end{pmatrix} + \begin{pmatrix} E \\ F \\ G \end{pmatrix} \times \begin{pmatrix} L_u \\ M_u \\ N_u \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} ha_v \\ hb_v \\ hc_v \end{pmatrix} = \begin{pmatrix} E_v \\ F_v \\ G_v \end{pmatrix} \times \begin{pmatrix} L \\ M \\ N \end{pmatrix} + \begin{pmatrix} E \\ F \\ G \end{pmatrix} \times \begin{pmatrix} L_v \\ M_v \\ N_v \end{pmatrix}. \tag{10}$$

Furthermore, we use the abbreviations

$$hd = hb^2 - 4 \cdot ha \cdot hc, \tag{11}$$

$$hd_u = 2 \cdot hb \cdot hb_u - 4 \cdot ha_u \cdot hc - 4 \cdot ha \cdot hc_u, \tag{12}$$

$$hd_v = 2 \cdot hb \cdot hb_v - 4 \cdot ha_v \cdot hc - 4 \cdot ha \cdot hc_v. \tag{13}$$

Then we can write two representatives of the solution classes of (7)—one for each class—in the form:

$$V_1 = \begin{pmatrix} vx_1 \\ vy_1 \end{pmatrix} = \begin{pmatrix} -2 \cdot ha + hb - \sqrt{hd} \\ 2 \cdot hc - hb - \sqrt{hd} \end{pmatrix}, \tag{14}$$

$$V_2 = \begin{pmatrix} vx_2 \\ vy_2 \end{pmatrix} = \begin{pmatrix} -2 \cdot ha + hb + \sqrt{hd} \\ 2 \cdot hc - hb + \sqrt{hd} \end{pmatrix}. \tag{15}$$

To show (14) and (15), we have to check (7) for

$$(vx, vy)^{\mathrm{T}} = (vx_1, vy_1)^{\mathrm{T}} \quad ((vx, vy)^{\mathrm{T}} = (vx_2, vy_2)^{\mathrm{T}} \text{ respectively})$$

and $(vx_1 \cdot \boldsymbol{x}_u + vy_1 \cdot \boldsymbol{x}_v) \cdot (vx_2 \cdot \boldsymbol{x}_u + vy_2 \cdot \boldsymbol{x}_v) = 0$. For the partial derivatives of V_1 and V_2 we obtain:

$$V_{1u} = \begin{pmatrix} -2 \cdot ha_u + hb_u - \dfrac{hd_u}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_u - hb_u - \dfrac{hd_u}{2 \cdot \sqrt{hd}} \end{pmatrix}, \tag{16}$$

$$V_{1v} = \begin{pmatrix} -2 \cdot ha_v + hb_v - \dfrac{hd_v}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_v - hb_v - \dfrac{hd_v}{2 \cdot \sqrt{hd}} \end{pmatrix}, \tag{17}$$

$$V_{2u} = \begin{pmatrix} -2 \cdot ha_u + hb_u + \dfrac{hd_u}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_u - hb_u + \dfrac{hd_u}{2 \cdot \sqrt{hd}} \end{pmatrix}, \tag{18}$$

$$V_{2v} = \begin{pmatrix} -2 \cdot ha_v + hb_v + \dfrac{hd_v}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_v - hb_v + \dfrac{hd_v}{2 \cdot \sqrt{hd}} \end{pmatrix}. \tag{19}$$

Inserting (14)–(19) into (4)–(6) gives the curvature of the lines of curvature for every surface point.

Critical points occur iff $V_1 = (0,0)^{\mathrm{T}}$ and $V_2 = (0,0)^{\mathrm{T}}$. Since

$$(vx_1 = 0) \wedge (vx_2 = 0) \Longleftrightarrow (hd = 0) \wedge (2 \cdot ha = hb),$$
$$(vy_1 = 0) \wedge (vy_2 = 0) \Longleftrightarrow (hd = 0) \wedge (2 \cdot hc = hb),$$

this is only possible for $2 \cdot ha = hb = 2 \cdot hc$. This and (8) give

$$0 = (ha, 2 \cdot ha, ha)^{\mathrm{T}} \cdot (E, F, G)^{\mathrm{T}} = ha \cdot (\boldsymbol{x}_u + \boldsymbol{x}_v)^2.$$

Since $\boldsymbol{x}$ is regularly parametrized, this is only possible for $ha = 0 = hb = hc$, i.e., we have an umbilical point. Therefore, lines of curvature produce critical points in (and only in) umbilical points on the surface.

Now we can formulate the desired

Theorem 1. *Given are two regularly parametrized and each C^3 continuous surfaces $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ which join along a common line l. Furthermore, every point on l is nonumbilical in $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$, and in no point of l the lines of curvature of $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are tangent to l. Then $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are G^3 along l iff their lines of curvature are G^2 across l.*

Proof. "$\Rightarrow$": If $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are G^3 along l they can be reparametrized in a way that they coincide in all partial derivatives of order $\leqslant 3$. Since the curvature formula of the lines of curvature contains only those derivatives (see Section 2), the lines of curvature are G^2.

"$\Leftarrow$": We assume that the junction line l is $(0,v), 0 \leqslant v \leqslant 1$. This can be done by a linear reparametrization of $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ without loss of generality.

The G^2 condition of the lines of curvature contains coincidence in surface normal, principal directions and principal curvatures, therefore G^2 of the surfaces along l. Thus, we can assume that $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are parametrized in such a way that

$$\begin{aligned} &\boldsymbol{x}(0,v)=\widetilde{\boldsymbol{x}}(0,v), \quad \boldsymbol{x}_u(0,v)=\widetilde{\boldsymbol{x}}_u(0,v), \quad \boldsymbol{x}_v(0,v)=\widetilde{\boldsymbol{x}}_v(0,v), \\ &\boldsymbol{x}_{uu}(0,v)=\widetilde{\boldsymbol{x}}_{uu}(0,v), \quad \boldsymbol{x}_{uv}(0,v)=\widetilde{\boldsymbol{x}}_{uv}(0,v), \quad \boldsymbol{x}_{vv}(0,v)=\widetilde{\boldsymbol{x}}_{vv}(0,v). \end{aligned} \tag{20}$$

From (20) and the assumption that l is the parametric line $u=0$ we obtain

$$\begin{aligned} &\boldsymbol{x}_{uuv}(0,v)=\widetilde{\boldsymbol{x}}_{uuv}(0,v), \quad \boldsymbol{x}_{uvv}(0,v)=\widetilde{\boldsymbol{x}}_{uvv}(0,v), \\ &\boldsymbol{x}_{vvv}(0,v)=\widetilde{\boldsymbol{x}}_{vvv}(0,v). \end{aligned} \tag{21}$$

Let $\dot{\boldsymbol{x}}_1$ and $\dot{\boldsymbol{x}}_2$ be the tangent vectors of the lines of curvature on $\boldsymbol{x}$. Furthermore, let $\dot{\widetilde{\boldsymbol{x}}}_1$ and $\dot{\widetilde{\boldsymbol{x}}}_2$ be the tangent vectors of the lines of curvature on $\widetilde{\boldsymbol{x}}$. Then (4), (14), (15) and (20) give

$$\dot{\boldsymbol{x}}_1(0,v)=(vx_1\cdot\boldsymbol{x}_u+vy_1\cdot\boldsymbol{x}_v)(0,v)=\dot{\widetilde{\boldsymbol{x}}}_1(0,v), \tag{22}$$

$$\dot{\boldsymbol{x}}_2(0,v)=(vx_2\cdot\boldsymbol{x}_u+vy_2\cdot\boldsymbol{x}_v)(0,v)=\dot{\widetilde{\boldsymbol{x}}}_2(0,v), \tag{23}$$

where vx_1, vy_1, vx_2, vy_2 are given by (14) and (15).

Let $\ddot{\boldsymbol{x}}_1$ and $\ddot{\boldsymbol{x}}_2$ be the second derivative vectors of the lines of curvature of $\boldsymbol{x}$, and let $\ddot{\widetilde{\boldsymbol{x}}}_1$ and $\ddot{\widetilde{\boldsymbol{x}}}_2$ be the second derivative vectors of the lines of curvature of $\widetilde{\boldsymbol{x}}$. Then (5), (14)–(19), (20) and (21) give

$$\ddot{\widetilde{\boldsymbol{x}}}_1(0,v)-\ddot{\boldsymbol{x}}_1(0,v)=vx_1\cdot\big(\boldsymbol{n}\cdot(\widetilde{\boldsymbol{x}}_{uuu}-\boldsymbol{x}_{uuu})\big)\cdot(a_1\cdot\boldsymbol{x}_u+b_1\cdot\boldsymbol{x}_v), \tag{24}$$

$$\ddot{\widetilde{\boldsymbol{x}}}_2(0,v)-\ddot{\boldsymbol{x}}_2(0,v)=vx_2\cdot\big(\boldsymbol{n}\cdot(\widetilde{\boldsymbol{x}}_{uuu}-\boldsymbol{x}_{uuu})\big)\cdot(a_2\cdot\boldsymbol{x}_u+b_2\cdot\boldsymbol{x}_v), \tag{25}$$

where

$$a_1=\frac{2\cdot ha\cdot F+hb\cdot G}{\sqrt{hd}}-G, \tag{26}$$

$$b_1=\frac{2\cdot ha\cdot F+hb\cdot G}{\sqrt{hd}}+2\cdot F+G, \tag{27}$$

$$a_2=\frac{2\cdot ha\cdot F+hb\cdot G}{\sqrt{hd}}+G, \tag{28}$$

$$b_2=\frac{2\cdot ha\cdot F+hb\cdot G}{\sqrt{hd}}-2\cdot F-G. \tag{29}$$

(The assumption that no umbilical point is on the junction line l ensures that $hd>0$ along l.)

Now the G^2 condition of the lines of curvature across l can be formulated in the following way (see (Farin, 1992, Chapter "Geometric Continuity I")):

$$\big(\ddot{\widetilde{\boldsymbol{x}}}_1(0,v)-\ddot{\boldsymbol{x}}_1(0,v)\big) \quad \text{parallel to } \dot{\boldsymbol{x}}_1(0,v), \tag{30}$$

$$\big(\ddot{\widetilde{\boldsymbol{x}}}_2(0,v)-\ddot{\boldsymbol{x}}_2(0,v)\big) \quad \text{parallel to } \dot{\boldsymbol{x}}_2(0,v). \tag{31}$$

Using (22), (23), (24), (25) and the fact that $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly independent, we can write (30) and (31) in the form

$$vx_1 \cdot \big(\boldsymbol{n} \cdot (\widetilde{\boldsymbol{x}}_{uuu} - \boldsymbol{x}_{uuu})\big) \cdot \det_1 = 0, \tag{32}$$

$$vx_2 \cdot \big(\boldsymbol{n} \cdot (\widetilde{\boldsymbol{x}}_{uuu} - \boldsymbol{x}_{uuu})\big) \cdot \det_2 = 0, \tag{33}$$

where

$$\det_1 = \det \begin{bmatrix} vx_1 & a_1 \\ vy_1 & b_1 \end{bmatrix}, \tag{34}$$

$$\det_2 = \det \begin{bmatrix} vx_2 & a_2 \\ vy_2 & b_2 \end{bmatrix}. \tag{35}$$

(22), (23) and the assumption that the lines of curvature are not parallel to l give

$$vx_1 \cdot vx_2 \neq 0. \tag{36}$$

From (26)–(29), (34) and (35) we obtain

$$\det_1 \cdot \det_2 = \frac{vx_1{}^2 \cdot vx_2{}^2 \cdot (F^2 - E \cdot G)}{hd}. \tag{37}$$

This, (36) and the assumption that $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are regularly parametrized yield

$$\det_1 \cdot \det_2 \neq 0. \tag{38}$$

From (32), (33), (36) and (38) we obtain

$$\big(\boldsymbol{n} \cdot (\widetilde{\boldsymbol{x}}_{uuu} - \boldsymbol{x}_{uuu})\big)(0, v) = 0. \tag{39}$$

Because of (39), there exist two scalar functions $r_1(v)$ and $r_2(v)$ so that

$$\widetilde{\boldsymbol{x}}_{uuu}(0, v) = \boldsymbol{x}_{uuu}(0, v) + r_1(v) \cdot \boldsymbol{x}_u(0, v) + r_2(v) \cdot \boldsymbol{x}_v(0, v). \tag{40}$$

Now we look for a reparametrization $\widehat{\boldsymbol{x}}$ of $\boldsymbol{x}$ which is C^3 to $\widetilde{\boldsymbol{x}}$ along l. We define

$$\widehat{\boldsymbol{x}}(u, v) = \boldsymbol{x}\big(\widehat{u}(u, v), \widehat{v}(u, v)\big), \tag{41}$$

$$\widehat{u}(u, v) = u + \tfrac{1}{6} \cdot u^3 \cdot r_1(v), \quad \widehat{v}(u, v) = v + \tfrac{1}{6} \cdot u^3 \cdot r_2(v).$$

Considering (41) to the junction line l (i.e. setting $u = 0$), we obtain:

$$\widehat{u}(0, v) = 0, \quad \widehat{u}_u(0, v) = 1, \quad \widehat{u}_{uu}(0, v) = 0, \quad \widehat{u}_{uuu}(0, v) = r_1(v), \tag{42}$$

$$\widehat{v}(0, v) = v, \quad \widehat{v}_u(0, v) = 0, \quad \widehat{v}_{uu}(0, v) = 0, \quad \widehat{v}_{uuu}(0, v) = r_2(v). \tag{43}$$

Applying the chain rule to (41), we obtain for the u-partials of $\widehat{\boldsymbol{x}}$:

$$\widehat{\boldsymbol{x}}_u = \widehat{u}_u \cdot \boldsymbol{x}_u + \widehat{v}_u \cdot \boldsymbol{x}_v, \tag{44}$$

$$\widehat{\boldsymbol{x}}_{uu} = \widehat{u}_u^2 \cdot \boldsymbol{x}_{uu} + 2 \cdot \widehat{u}_u \cdot \widehat{v}_u \cdot \boldsymbol{x}_{uv} + \widehat{v}_u^2 \cdot \boldsymbol{x}_{vv} + \widehat{u}_{uu} \cdot \boldsymbol{x}_u + \widehat{v}_{uu} \cdot \boldsymbol{x}_v, \tag{45}$$

$$\begin{aligned}\widehat{\boldsymbol{x}}_{uuu} = {} & \widehat{u}_u^3 \cdot \boldsymbol{x}_{uuu} + 3 \cdot \widehat{u}_u^2 \cdot \widehat{v}_u \cdot \boldsymbol{x}_{uuv} + 3 \cdot \widehat{u}_u \cdot \widehat{v}_u^2 \cdot \boldsymbol{x}_{uvv} + \widehat{v}_u^3 \cdot \boldsymbol{x}_{vvv} \\ & + 3 \cdot \big(\widehat{u}_u \cdot \widehat{u}_{uu} \cdot \boldsymbol{x}_{uu} + (\widehat{v}_u \cdot \widehat{u}_{uu} + \widehat{u}_u \cdot \widehat{v}_{uu}) \cdot \boldsymbol{x}_{uv} + \widehat{v}_u \cdot \widehat{v}_{uu} \cdot \boldsymbol{x}_{vv}\big) \\ & + \widehat{u}_{uuu} \cdot \boldsymbol{x}_u + \widehat{v}_{uuu} \cdot \boldsymbol{x}_v. \end{aligned} \tag{46}$$

Setting $u = 0$, we obtain from (44)–(46) using (42) and (43):

$$\widehat{\boldsymbol{x}}(0,v) = \boldsymbol{x}(0,v) = \widetilde{\boldsymbol{x}}(0,v), \tag{47}$$

$$\widehat{\boldsymbol{x}}_u(0,v) = \boldsymbol{x}_u(0,v) = \widetilde{\boldsymbol{x}}_u(0,v), \tag{48}$$

$$\widehat{\boldsymbol{x}}_{uu}(0,v) = \boldsymbol{x}_{uu}(0,v) = \widetilde{\boldsymbol{x}}_{uu}(0,v), \tag{49}$$

$$\begin{aligned}\widehat{\boldsymbol{x}}_{uuu}(0,v) &= \boldsymbol{x}_{uuu}(0,v) + r_1(v)\cdot\boldsymbol{x}_u(0,v) + r_2(v)\cdot\boldsymbol{x}_v(0,v),\\ &= \widetilde{\boldsymbol{x}}_{uuu}(0,v).\end{aligned} \tag{50}$$

From (47)–(50) we obtain

$$\begin{aligned}&\widehat{\boldsymbol{x}}_v(0,v) = \widetilde{\boldsymbol{x}}_v(0,v), \quad \widehat{\boldsymbol{x}}_{uv}(0,v) = \widetilde{\boldsymbol{x}}_{uv}(0,v),\\ &\widehat{\boldsymbol{x}}_{vv}(0,v) = \widetilde{\boldsymbol{x}}_{vv}(0,v), \quad \widehat{\boldsymbol{x}}_{uuv}(0,v) = \widetilde{\boldsymbol{x}}_{uuv}(0,v),\\ &\widehat{\boldsymbol{x}}_{uvv}(0,v) = \widetilde{\boldsymbol{x}}_{uvv}(0,v), \quad \widehat{\boldsymbol{x}}_{vvv}(0,v) = \widetilde{\boldsymbol{x}}_{vvv}(0,v).\end{aligned} \tag{51}$$

Therefore, $\widehat{\boldsymbol{x}}$ and $\widetilde{\boldsymbol{x}}$ are C^3 along l, which gives that $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are G^3 along l. □

Remark 1. If there is only a single point $\boldsymbol{x}_s$ on the junction line l which is umbilic or in which one of the lines of curvature is tangent to l, this point $\boldsymbol{x}_s$ divides l in two parts which both (except for $\boldsymbol{x}_s$ itself) fulfill Theorem 1. Since $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are continuous, we still can infer G^3 of the surface from G^2 of the lines of curvature across $l \setminus \{\boldsymbol{x}_s\}$.

Remark 2. The proof of Theorem 1 used the assumption that *both* lines of curvature are G^2 across l only for making sure that $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are G^2 along l. Therefore, we can rewrite Theorem 1 in the following form:

Given are two regularly parametrized and each C^3 continuous surfaces $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ which are G^2 along a common line l. Furthermore, every point on l is nonumbilical in $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$, and in no point of l the lines of curvature of $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are tangent or perpendicular to l. Then $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are G^3 along l iff there is one family of lines of curvature which is G^2 across l.

Remark 3. Theorem 1 has some similarities to the *linkage curve theorem* described in (Pegna and Wolter, 1992). In this theorem, the sufficient condition for G^2 of two surfaces along a common line l is the continuity of the normal curvature of a family of surface curves across l. That means, both Theorem 1 and the linkage curve theorem use curvature properties of families of curves across the junction line to obtain conditions for geometric continuity of the surface.

Remark 4. A similar theorem to Theorem 1 can be formulated for asymptotic lines:

Given are two regularly parametrized and each C^3 continuous surfaces $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ which join along a common line l. Furthermore, every point on l has negative Gaussian curvature in $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$, and in no point of l is one of the asymptotic lines of $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ tangent to l. Then $\boldsymbol{x}$ and $\widetilde{\boldsymbol{x}}$ are G^3 along l iff their asymptotic lines are G^2 across l.

See (Theisel, 1996) for a proof.

4. Visualizing the curvature of the lines of curvature

Since we have shown how to compute the curvature of the lines of curvature, we can use this for visualizing certain surface properties.

The upper left picture of Fig. 1 shows a ray traced test surface: a shoe shaped (non-rational) piecewise bicubic surface. It consists of 29×10 patches and is G^2 along the patch borders.

We compute the geodesic curvature of the lines of curvature for every surface point and color code those values. The geodesic curvature can be obtained by projecting $\ddot{x}$ into the tangent plane and applying (6). The curvature obtained this way can be considered as the curvature of a 2D curve and therefore equipped with a sign.

Fig. 1. Test surface and geodesic curvature of its lines of curvature.

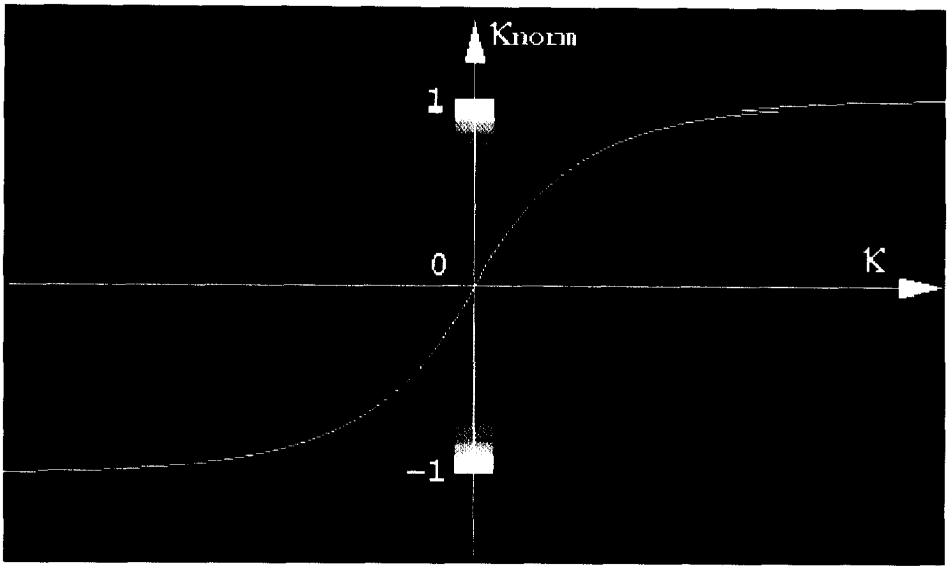

Fig. 2. Color coding the geodesic curvature.

The color coding map for the geodesic curvature of the lines of curvature is shown in Fig. 2: The curvature κ (which can lie anywhere between $-\infty$ and ∞) is "normalized" to κ_{norm} in the interval $\langle -1, 1\rangle$ using the equation

$$\kappa_{\mathrm{norm}} = \operatorname{sgn}(\kappa) \cdot \left(1 - \mathrm{e}^{-\|\kappa\| \cdot \mathrm{con}}\right).$$

The positive value con can be considered as the contrast of the visualization. Decreasing con leads to a darker picture but emphasizes the critical points (in this case: the umbilical points). con should be chosen interactively.

The pictures middle left and middle right of Fig. 1 show the geodesic curvature of the two families of lines of curvature. The lower two pictures are magnifications of the middle ones. For comparison, the upper right picture shows the visualization of the Gaussian curvature of the test surface.

The curvature visualizations of the lines of curvature show clearly discontinuities at the patch borders of the test surface. This shows that the surface is not G^3 continuous. The umbilical points of the surface can be clearly detected as highlights: around umbilical points the curvature of the lines of curvature tends to infinity.

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