The Curvature of Characteristic Curves on Surfaces

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Analyzing and interrogating designed surfaces remains an important and widely researched issue in computer-aided geometric design (CAGD). Treating families of characteristic curves—such as contour lines, lines of curvature, asymptotic lines, isophotes, and reflection lines—on the surface proves a popular method of doing this. All these curves have something in common:

- They reflect the surface's geometric properties. Analyzing these curves gives information about shape and continuity of the surface. The curves do not depend on the parametrization of the surface.
- For sufficiently complicated surfaces (such as bicubic polynomial surfaces), these curves can only be described as the solutions of differential equations. Working with the curves themselves requires solving those equations numerically.

We show how to compute the curvature and geodesic curvature of characteristic curves on surfaces, such as contour lines, lines of curvature, asymptotic lines, isophotes, and reflection lines.

In this article we want to use the power of the characteristic curves, but avoid the latter problem described above. We achieve this by considering not the curves directly, but one of their most important properties: their *curvatures*. Although we do not have explicit formulas for these curves, we can express their curvatures and geodesic curvatures. For particular curves, we introduce their thickness as another characteristic property. Finally, we discuss how to use the visualization of the characteristic curves' curvatures as a surface interrogation tool.

Theoretical background

We will be analyzing a family of curves on a parametric surface by interpreting it as tangent curves of vector fields. Before we discuss the surface case, we briefly describe the case of 2D vector fields.

Tangent curves of vector fields

Given is a 2D vector field $V: \mathbb{E}^2 \to \mathbb{R}^2$. V assigns a vector $(vx(P), vy(P))^T$ to any point $P \sim (u, v)$. We use the notation $V(P) = V(u, v) = (vx(u, v), vy(u, v))^T$. A point $P \in \mathbb{E}^2$ is called a *critical point of V* if $V(P) = 0$ is the zero vector. A curve $L \subseteq \mathbb{E}^2$ is called a tangent curve (for example, stream line, flow line, or characteristic curve) of the vector field V if the following condition is satisfied: For all points $P \in L$, the tangent vector of the curve at the point P has the same direction as the vector $V(P)$. For every point $P \in \mathbb{E}^2$ there is one and only one tangent curve through it (except for critical points of V). Tangent curves do not intersect each other (except for critical points of V). The vector field may be interpreted as representing a 2D flow. The tangent curves are simply the trajectories of particles moving in the flow. See Davis[1] for a survey on vector analysis.

Given a noncritical point $P_0 \sim (u_0, v_0)$ in the vector field, we want to compute the curvature of the tangent curve through P_0 at the point P_0. Let $L(t)$ be the tangent curve through P_0 and $L(t_0) = P_0$. From the definition of the tangent curves, we know about the first derivative vector of L:

$$\dot{L}(t_0) = V\left(L(t_0)\right) = V(P_0) \tag{1}$$

Applying the chain rule to Equation 1, we obtain for the second derivative vector of L

$$\begin{aligned}\ddot{L}(t_0) &= V_u \cdot \frac{du}{dt}(t_0) + V_v \cdot \frac{dv}{dt}(t_0) \\ &= \left(vx \cdot V_u + vy \cdot V_v\right)(P_0)\end{aligned} \tag{2}$$

Using Equations 1 and 2, we can compute the signed curvature of the tangent curve through P_0:

$$\kappa(t_0) = \frac{\det\left[\dot{L}(t_0), \ddot{L}(t_0)\right]}{\left\|\dot{L}(t_0)\right\|^3} \tag{3}$$

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From Equations 1, 2, and 3 we can compute the curvature of the tangent curve for every point of the vector field:

$$\kappa(V) = \frac{vx \cdot \det[V, V_u] + vy \cdot \det[V, V_v]}{\|V\|^3} \quad (4)$$

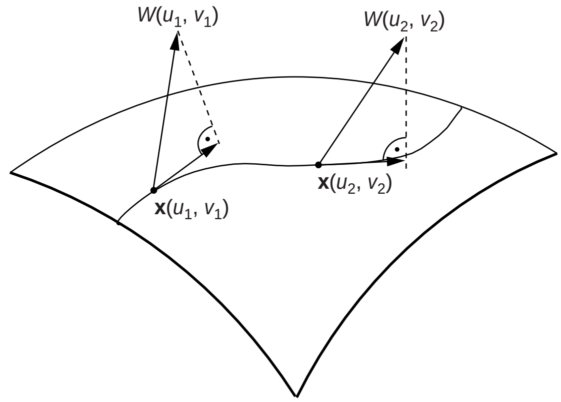
The curvature of tangent curves is only defined for noncritical points of the vector field. Further properties of $\kappa(V)$ are discussed elsewhere.[2]

Tangent curves on surfaces

Tangent curves on surfaces can be defined in two ways:

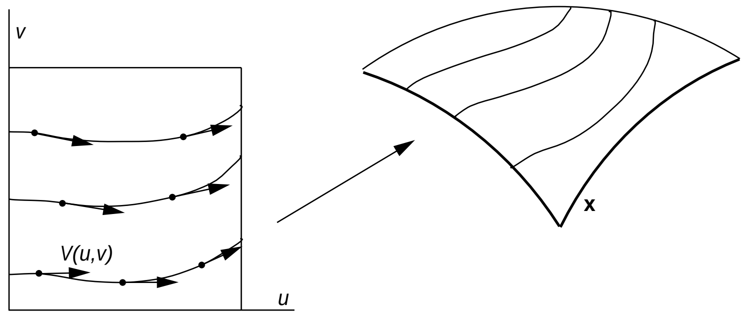
Definition 1. Given is a surface $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ and a map $W : \mathbb{E}^2 \to \mathbb{R}^3$. W relates any point of the domain of $\mathbf{x}$ to a vector in 3D. Together with $\mathbf{x}$, W can be considered as relating any point $\mathbf{x}(u, v)$ on the surface to a 3D vector $W(u, v)$. W is called a vector field over the surface $\mathbf{x}$.

A tangent curve defined by W is a curve on the surface where the tangent vector and the projection of $W(u, v)$ into the tangent plane of $\mathbf{x}(u, v)$ have the same direction for every point of the curve. Figure 1 illustrates this definition.



1 **A vector field *W* over a surface x. Shown are x, *W*, and the projection of *W* in the tangent plane of x for two points (u_1, v_1) and (u_2, v_2).**

Definition 2. Given is a surface $\mathbf{x}$ and a 2D vector field V in the domain of $\mathbf{x}$. V produces a family of tangent curves in the domain. The maps of these domain curves onto the surface $\mathbf{x}$ are called tangent curves on the surface $\mathbf{x}$. Figure 2 illustrates this definition.



2 **A vector field *V* in the domain and the mapping of its tangent curves onto the surface x.**

We discuss the correlation between these definitions in the section "Corresponding vector fields on surfaces."

Throughout this article, we use the following notations and abbreviations:

$$\dot{\mathbf{x}}(u, v) \text{ and } \ddot{\mathbf{x}}(u, v)$$

denote the first and second derivative vector of the tangent curve on $\mathbf{x}$ through $\mathbf{x}(u, v)$ in the point $\mathbf{x}(u, v)$. (Since we deal with corresponding vector fields W and V in the following, we do not distinguish in the notation of $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ whether they are obtained from W or V.)

$$\mathbf{n} = \mathbf{n}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

is the *normalized normal vector* of the surface $\mathbf{x}(u, v)$.

Furthermore, we use the classical abbreviations

$E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$, $G = \mathbf{x}_v \cdot \mathbf{x}_v$
$L = \mathbf{n} \cdot \mathbf{x}_{uu}$, $M = \mathbf{n} \cdot \mathbf{x}_{uv}$, $N = \mathbf{n} \cdot \mathbf{x}_{vv}$

Corresponding vector fields on surfaces

A family of smooth curves on a parametric surface is associated with a corresponding family of pre-image curves in the surface's domain. This family of curves is associated with a 2D vector field, namely the vector field of its derivatives. Instead of dealing with the surface curves directly, we relate them to this domain vector field.

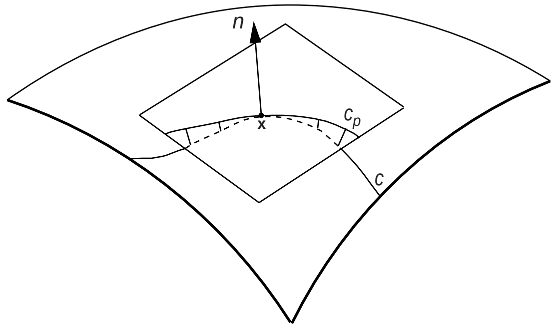
Definition 3. Given is a surface $\mathbf{x}$, a 2D vector field V in the domain of $\mathbf{x}$ and a 3D vector field W over $\mathbf{x}$. W and V are called corresponding referring to the surface $\mathbf{x}$, if the two families of tangent curves obtained from W (using definition 1) and V (using definition 2) are identical.

Definition 3 provides a reason to solve the following two problems:

1. Given is a surface $\mathbf{x}$ and a domain vector field V. We look for a corresponding vector field W over $\mathbf{x}$ as well as for the derivatives $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ of the tangent curves on $\mathbf{x}$.
2. Given is a surface $\mathbf{x}$ and a vector field W over $\mathbf{x}$. We look for a corresponding domain vector field V, for $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$.

We start with problem 1. Given a curve $[u = u(t), v = v(t)]$ in the domain, we can easily compute the map of this curve on the surface and its first and second derivative vector (see Farin[3] for further discussion):

$$\mathbf{x}(t) = \mathbf{x}(u(t), v(t))$$
$$\dot{\mathbf{x}}(t) = (\dot{u} \cdot \mathbf{x}_u + \dot{v} \cdot \mathbf{x}_v)(t)$$
$$\ddot{\mathbf{x}}(t) = \begin{pmatrix} \ddot{u} \cdot \mathbf{x}_u + \ddot{v} \cdot \mathbf{x}_v + \dot{u}^2 \cdot \mathbf{x}_{uu} \\ + 2 \cdot \dot{u} \cdot \dot{v} \cdot \mathbf{x}_{uv} + \dot{v}^2 \cdot \mathbf{x}_{vv} \end{pmatrix}(t) \quad (5)$$



3 **Geodesic curvature of a surface curve *c*: c_p is the projection of *c* into the tangent plane of x = x(*u*, *v*). The geodesic curvature of *c* in x(*u*, *v*) is the curvature of c_p in x(*u*, *v*).**

Considering the domain curve as a tangent curve in the domain defined by V, we know the first and the second derivative vector in any point of the domain from Equations 1 and 2:

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = V,\ \begin{pmatrix}\ddot{u}\\ \ddot{v}\end{pmatrix} = vx \cdot V_u + vy \cdot V_v \qquad (6)$$

A corresponding vector field to V is $W = \dot{\mathbf{x}}$.

Moving to problem 2, given the vector field W over $\mathbf{x}$, the first derivative vector of the tangent curve is the projection of W onto the tangent plane for every point of the surface

$$\dot{\mathbf{x}} = W - (W \cdot \mathbf{n}) \cdot \mathbf{n} \qquad (7)$$

To obtain a corresponding vector field V, we have to solve the linear system of equations

$$\dot{\mathbf{x}} = vx \cdot \mathbf{x}_u + vy \cdot \mathbf{x}_v$$

This system consists of three equations and the two unknowns vx and vy. Since $\dot{\mathbf{x}}$, $\mathbf{x}_u$, and $\mathbf{x}_v$ are all in one plane (namely in the tangent plane of $\mathbf{x}$), we can find a solution for any regularly parametrized surface:

$$V = \begin{pmatrix} vx \\ vy \end{pmatrix} = \frac{1}{\left\| \mathbf{x}_u \times \mathbf{x}_v \right\|} \cdot \begin{pmatrix} \det[\mathbf{n}, W, \mathbf{x}_v] \\ -\det[\mathbf{n}, W, \mathbf{x}_u] \end{pmatrix} \qquad (8)$$

Equation 8 has reduced problem 2 to problem 1.

Since we know $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ for every point on $\mathbf{x}$, we can compute the curvature κ of the tangent curves on the surface:

$$\kappa = \frac{\left\| \dot{\mathbf{x}} \times \ddot{\mathbf{x}} \right\|}{\left\| \dot{\mathbf{x}} \right\|^3} \qquad (9)$$

The geodesic curvature κ_g of a surface curve can be obtained by projecting the curve into the tangent plane and computing the curvature of this projected curve. See Figure 3 for an illustration.

Since the geodesic curvature can be interpreted as the curvature of a 2D curve, we can equip it with a sign. This sign gives additional information about the surface curves. We obtain the following equation for the geodesic curvature:

$$\kappa_g = \operatorname{sign}\left(\det[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \mathbf{n}]\right) \cdot \frac{\left\| \dot{\mathbf{x}} \times \ddot{\mathbf{x}}_g \right\|}{\left\| \dot{\mathbf{x}} \right\|^3} \qquad (10)$$

where

$$\ddot{\mathbf{x}}_g = \ddot{\mathbf{x}} - (\ddot{\mathbf{x}} \cdot \mathbf{n}) \cdot \mathbf{n}$$

is the projection of $\ddot{\mathbf{x}}$ into the tangent plane.

Thickness of tangent curves

We consider the special case that a scalar field $s(u, v)$ over $\mathbf{x}$ is given and the desired tangent curves are the equipotential lines of s on the surface. For this special case there is a simple way to draw a representation of these curves: Mark all points on the surface that have a value of s in a fixed (small) interval. The resulting lines on the surface are actually point sets with a changing thickness or distance between adjacent lines. This thickness can provide information about the behavior of the surface and the tangent curves.

The thickness of a tangent curve on a surface $\mathbf{x}(u, v)$ denotes how rapidly the values of the scalar field change around $\mathbf{x}(u, v)$. A thin tangent curve indicates a strong change of the values of s around $\mathbf{x}(u, v)$.

We consider the scalar field s in the surface's domain. A measure of how much s changes around a point (u, v) is the magnitude of the gradient $(s_u, s_v)^T$. We only have to map this gradient vector onto the surface in an appropriate form. The magnitude of the resulting vector on the surface tells us how much s changes on the surface around the point $\mathbf{x}(u, v)$.

The projection $\mathbf{x}_{gr}$ of the gradient vector—or to be precise, a vector perpendicular to the gradient vector but with the same magnitude—has the following equation:

$$\mathbf{x}_{gr} = \frac{-s_v \cdot \mathbf{x}_u + s_u \cdot \mathbf{x}_v}{\left\| \mathbf{x}_u \times \mathbf{x}_v \right\|} \qquad (11)$$

Then the thickness $th(u, v)$ of the tangent curve through $\mathbf{x}(u, v)$ can be expressed in the form

$$th = \frac{1}{\left\| \mathbf{x}_{gr} \right\|} \qquad (12)$$

Equations 11 and 12 show that th around a critical point tends to infinity.

Particular tangent curves on surfaces

Now we want to apply the results from the section "Theoretical background" to particular tangent curves on surfaces, namely contour lines, lines of curvature, asymptotic lines, isophotes, and reflection lines. To do this, we only have to show how to compute the domain vector field V (which corresponds to the tangent curves) and its partial derivatives.

Contour lines

A family of contour lines is defined by a normalized direction vector $\mathbf{r}$ in the 3D space. We consider all planes perpendicular to $\mathbf{r}$. The intersections of these planes with the surface yield a family of curves on the surface—the contour lines. Therefore, points on the surface are located on the same contour line referring to $\mathbf{r}$ if the scalar field

$$s(u, v) = \mathbf{r} \cdot (\mathbf{x} - (0, 0, 0)^T) \quad \textbf{(13)}$$

gives the same values for those points. Thus, contour lines are the equipotential lines of the scalar field s. The direction of these lines in the domain can be computed as the perpendiculars to the gradients. To obtain the directions of the contour lines in the domain, we compute

$$V = \begin{pmatrix} -s_v \\ s_u \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot \mathbf{x}_v \\ \mathbf{r} \cdot \mathbf{x}_u \end{pmatrix} \quad \textbf{(14)}$$

which gives for the partial derivatives

$$V_u = \begin{pmatrix} -s_{uv} \\ s_{uu} \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot \mathbf{x}_{uv} \\ \mathbf{r} \cdot \mathbf{x}_{uu} \end{pmatrix}, V_v = \begin{pmatrix} -s_{vv} \\ s_{uv} \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot \mathbf{x}_{vv} \\ \mathbf{r} \cdot \mathbf{x}_{uv} \end{pmatrix} \quad \textbf{(15)}$$

The vector field corresponding to contour lines has a critical point iff $\mathbf{n}$ and $\mathbf{r}$ have the same direction. Contour lines across a surface's patch boundaries are G^2 continuous if the surface is G^2 along the patch boundaries.[4] Given s, V, and their partial derivatives, we can compute the contour lines' curvature, geodesic curvature, and thickness for every point on the surface (except for critical points).

Lines of curvature

Lines of curvature are the tangent curves of the principal directions. They're considered as a vector field on the surface. Since two principal direction vector fields (whose vectors remain perpendicular to each other) exist, we have two families of lines of curvature.

The principal directions in the domain of $\mathbf{x}$ are the solutions $(vx, vy)^T$ of the quadratic equation:[3]

$$\det \begin{bmatrix} vy^2 & -vx \cdot vy & vx^2 \\ E & F & G \\ L & M & N \end{bmatrix} = 0 \quad \textbf{(16)}$$

Equation 16 yields two solution classes of $(vx, vy)^T$ (where a solution class contains only vectors of the same direction). We use the abbrevations ha, hb, hc, and hd, which are defined as

$$\begin{pmatrix} ha \\ hb \\ hc \end{pmatrix} = \begin{pmatrix} E \\ F \\ G \end{pmatrix} \times \begin{pmatrix} L \\ M \\ N \end{pmatrix}$$

$$hd = hb^2 - 4 \cdot ha \cdot hc$$

Then we can write two representatives of Equation 16's solution classes—one for each class—in the form:

$$V_1 = \begin{pmatrix} vx_1 \\ vy_1 \end{pmatrix} = \begin{pmatrix} -2 \cdot ha + hb - \sqrt{hd} \\ 2 \cdot hc - hb - \sqrt{hd} \end{pmatrix} \quad \textbf{(17)}$$

$$V_2 = \begin{pmatrix} vx_2 \\ vy_2 \end{pmatrix} = \begin{pmatrix} -2 \cdot ha + hb + \sqrt{hd} \\ 2 \cdot hc - hb + \sqrt{hd} \end{pmatrix} \quad \textbf{(18)}$$

Applying basic differentiation rules gives the partial derivatives of V_1 and V_2. Critical points have $V_1 = (0, 0)^T$ and $V_2 = (0, 0)^T$. This occurs in (and only in) umbilical points of the surface.[2] Lines of curvature across the patch boundaries remain G^2 continuous if the surface is G^3 at the patch boundaries.[3]

Asymptotic lines

Asymptotic lines are defined by the vector field $(vx, vy)^T$ that satisfies[3]

$$L \cdot vx^2 + 2 \cdot M \cdot vx \cdot vy + N \cdot vy^2 = 0 \quad \textbf{(19)}$$

We have two real solution classes for a negative Gaussian curvature, one solution class for a zero Gaussian curvature, and only complex solutions for a positive Gaussian curvature.[3] For a negative Gaussian curvature, the directions of the surface's asymptotic lines coincide with the directions of the Dupin's indicatrices (in this case a pair of hyperbolae). The defining geometric property of asymptotic lines is a zero normal curvature in every point of the surface. Here we only consider the case of a negative Gaussian curvature. We obtain the following as representatives of the two solution classes:

$$V_1 = \begin{pmatrix} N - M - \sqrt{M^2 - L \cdot N} \\ L - M + \sqrt{M^2 - L \cdot N} \end{pmatrix}$$

$$V_2 = \begin{pmatrix} N - M + \sqrt{M^2 - L \cdot N} \\ L - M - \sqrt{M^2 - L \cdot N} \end{pmatrix}$$

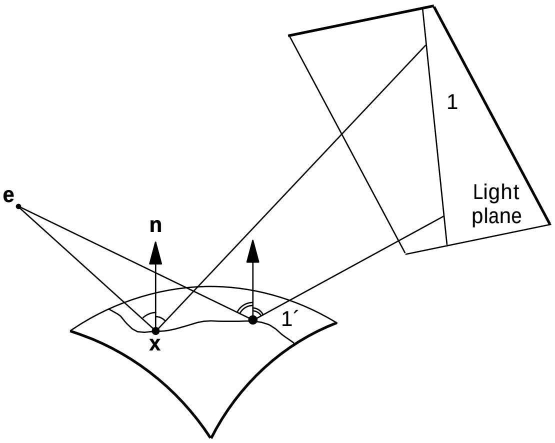
Critical points have $V_1 = (0, 0)^T$ and $V_2 = (0, 0)^T$. This does not occur in areas of a negative Gaussian curvature.[2] Asymptotic lines across a surface's patch boundaries are G^2 continuous if the surface is G^3 at the patch boundaries.

Isophotes

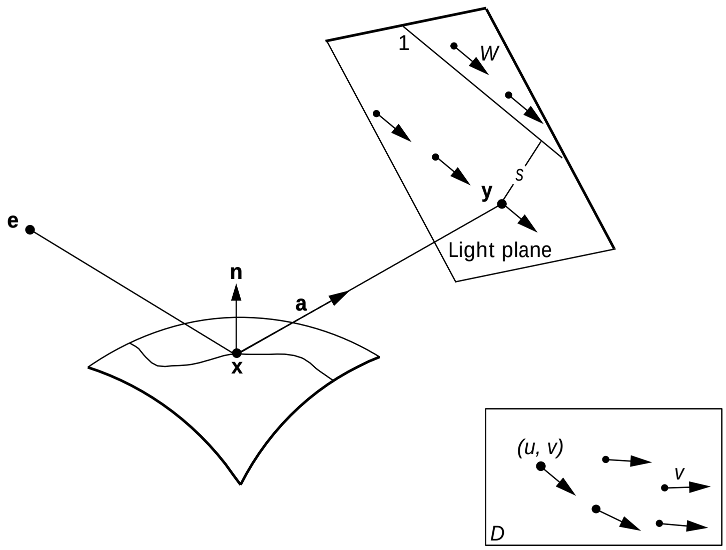
Poeschl[5] discussed isophotes as a surface interrogation tool. We define a family of isophotes by an eye point $\mathbf{e} = (ex, ey, ez)^T$. The isophotes then become equipotential lines of the scalar field

$$s(u, v) = \frac{(\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}}{\|\mathbf{e} - \mathbf{x}\|} \quad \textbf{(20)}$$

on the surface. This means an isophote on a surface con-



4 Reflection line l' on the surface x: l' is the mirror image of the straight line l.



5 Reflection lines—definition of the surface y and the corresponding vector fields W and V.

tains all surface points that have the same angle between the eye vector (that is, eye point minus surface point) and the surface normal **n**. Silhouette lines are the special case $s = 0$ of isophotes.

Similar to contour lines, we obtain the following for the domain vector field:

$$V = \begin{pmatrix} -s_v \\ s_u \end{pmatrix}, \; V_u = \begin{pmatrix} -s_{uv} \\ s_{uu} \end{pmatrix}, \; V_v = \begin{pmatrix} -s_{vv} \\ s_{uv} \end{pmatrix} \qquad \textbf{(21)}$$

which yields the isophotes' curvature, geodesic curvature, and thickness.

The appearance of critical points depends on the choice of **e**. In fact, for every surface point $\mathbf{x}_0$ we can find an appropriate eye point so that the isophotes have a critical point in $\mathbf{x}_0$. Therefore, critical points do not directly provide information about the surface's behavior.

Isophotes across patch boundaries on the surface remain G^2 continuous if the surface is G^3 at the patch boundaries.[6]

Reflection lines

Reflection lines[7,8] are a standard surface interrogation tool in car design. They can tell the viewer much about a surface's aesthetic qualities. Given is a surface **x**, an eye point **e**, a plane, and a family of parallel straight lines in the plane. The plane is called the light plane. The surface **x** is considered mirror-like. Reflection lines on the surface **x** are the mirror image of the family of straight lines in the plane while looking from the eye point **e** (see Figure 4).

The definition of reflection lines depends on a particular configuration. This configuration contains the eye point's location, the light plane, and the direction of the lines in the light plane. We want to compute the curvature of the reflection lines for a given surface and a given configuration.

Let D be the domain of the surface **x**. We define another surface **y** over D in the following way: For every point $(u, v) \in D$, we take $\mathbf{x}(u, v)$, compute the surface normal $\mathbf{n}(u, v)$ in $\mathbf{x}(u, v)$, compute the reflected ray **a** of $\mathbf{x} - \mathbf{e}$ in the tangent plane of $\mathbf{x}(u, v)$, and intersect this ray with the light plane given in the configuration. The intersection point of **a** and the light plane is considered as $\mathbf{y}(u, v)$ (see Figure 5).

The surface **y** lies completely in the light plane. Therefore, all partial derivative vectors of **y** remain in the light plane as well. Since we know the surface and the configuration, we can deduce the formula of **y** and its partial derivatives.

We consider a vector field W over **y**, which has for every point the direction of the parallel straight lines in the light plane. This constant vector field produces the family of straight lines as tangent curves. We compute the vector field V in the domain that corresponds to W over **y** (see Figure 5). V and its partial derivatives give the curvature and the geodesic curvature of the reflection lines. Theisel[2] provides a detailed description of the formulas of **y**, V, and their partial derivatives.

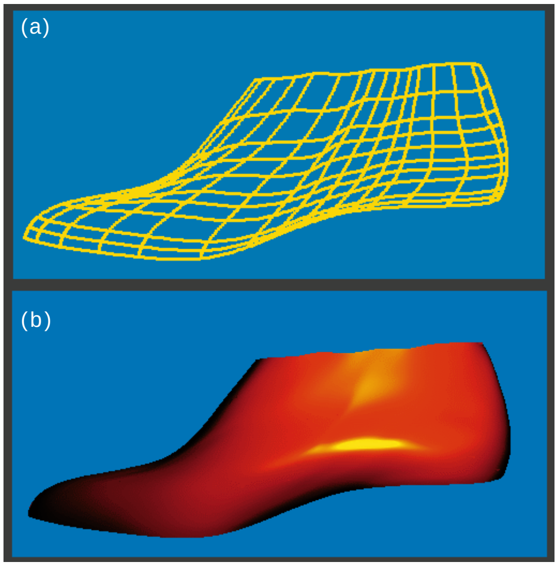
The thickness of reflection lines has a nice practical meaning—when analyzing a surface using reflection lines, a designer moves relative to the surface and observes how quickly the reflection lines move on the surface. Thus, the thickness becomes a measure of how quickly the reflection lines move.

To compute the thickness of the reflection lines, we have to find a scalar field s so that the reflection lines are the equipotential lines of s. We define a fixed straight line l in the light plane with the direction of W (see Figure 5). Then

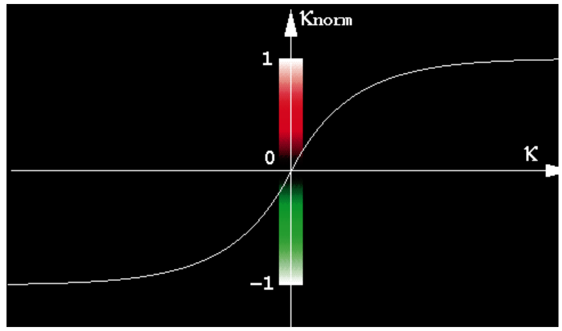
$$s = \mathrm{dist}(\mathbf{y}, l) \qquad \textbf{(22)}$$

is the scalar field defining reflection lines. From Equation 22 we can compute the thickness similar to the case of contour lines and isophotes.

The appearance of critical points depends on the par-



6 **Test surface—(a) line drawing of the patch boundaries and (b) ray-traced image.**



7 **Color coding the curvature of vector fields.**

ticular configuration and thus does not provide direct information about the surface. Reflection lines across patch boundaries on the surface remain G^2 continuous if the surface is G^3 at the patch boundaries.

Tangent curves and surface interrogation

A surface designed by a CAD system may look perfect in the wireframe (and even in the shaded) representation. Nevertheless, the surface can have imperfections, an undesired behavior of characteristic properties, or areas that simply do not look nice. Surface interrogation algorithms point out these imperfections on the surface.

Surface interrogation algorithms focus on the following two points:

1. They point out geometric properties of the surface that can usually be described in mathematical terms. These properties can be geometric continuity at the patch boundaries, special points on the surface (flat points, umbilical points), convexity properties (change of the sign of the curvatures), and strong and frequent variations of the curvatures.
2. They give a global impression about smoothness and fairness of the surface.

A variety of surface interrogation algorithms have been developed that emphasize different aspects of these points. Hagen et al.[6] and Hahmann[9] presented surveys on such surface interrogation algorithms.

The tangent curves on surfaces discussed in the section "Particular tangent curves on surfaces" are well known as standard surface interrogation tools. For example, Beck[10] discussed using contour lines and lines of curvature. In addition, Poeschl[5] discussed isophotes, while Klass[7] and Kaufmann[8] discussed reflection lines.

It is also a standard method of surface interrogation to visualize surface properties by color coding them on the surface. Beck[10] and Dill[11] have visualized the Gaussian and the mean curvature of a surface.

In this section, we want to discuss the additional use of the curvature plots and, if possible, the thickness plots of the characteristic curves on the surface as surface interrogation tools. This gives more information about the surface's geometric properties. In fact, the usual application of those characteristic curves gives us only first and second order information (that is, properties that contain only the first and second partial derivatives of the surface). Also, the visualization of Gaussian and mean curvatures covers only first and second order information of the surface.

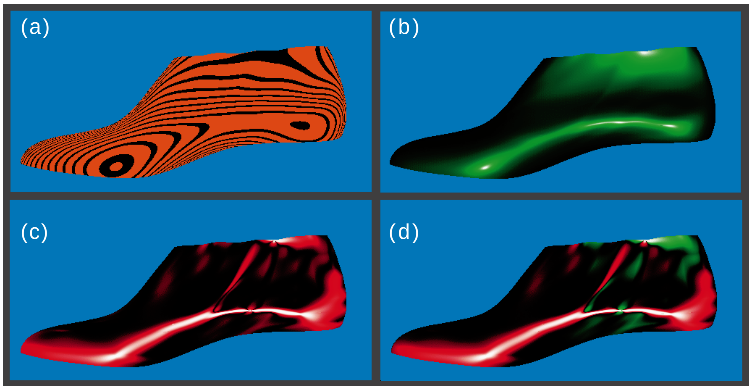
Visualizations of the curvatures of lines of curvature, asymptotic lines, isophotes, and reflection lines gives additional third order information about the surface.

The test surface. Figure 6 shows the test surface—a shoe-shaped (nonrational) bicubic B-spline surface. This surface consists of 15 × 10 patches and is C^2 continuous at the patch borders. The top of Figure 6 shows a line drawing of the patch boundaries. The bottom shows the ray-traced image. For this picture, one light source was used in the same position as the eye point. We can hardly see any imperfections in these images of the surface.

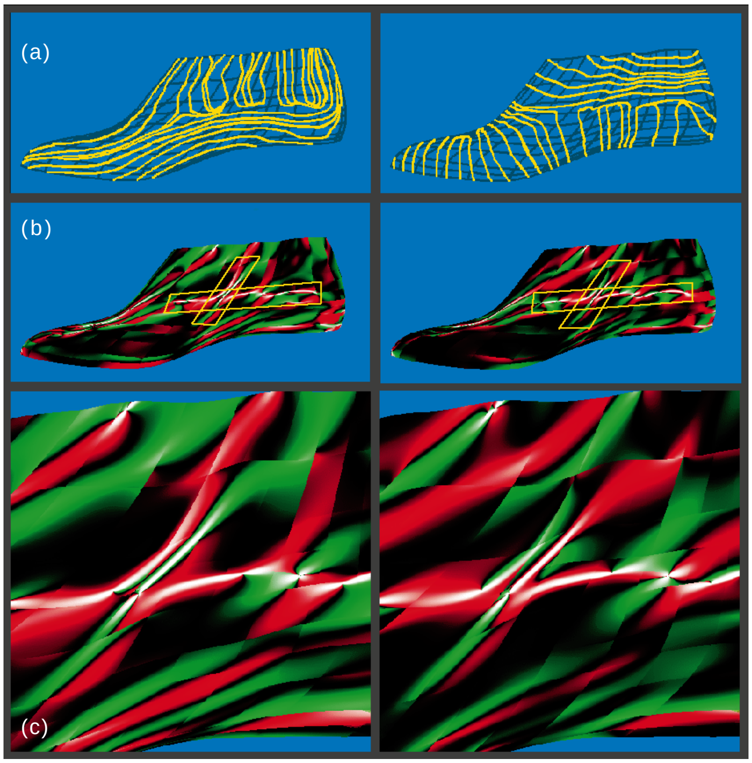
Curvature and thickness for surface interrogation. We compute the curvature, geodesic curvature, and (if possible) thickness of the tangent curves for an appropriate number of surface points and color code these values. To do this, we use a color-coding map, which can be obtained in the following way: the curvature κ (which can lie anywhere between $-\infty$ and ∞) is normalized to κ_{norm} in the interval $\langle -1, 1 \rangle$ using the equation

$$\kappa_{norm} = \operatorname{sgn}(\kappa) \cdot \left(1 - e^{-\|\kappa\| \cdot con}\right) \qquad \textbf{(23)}$$

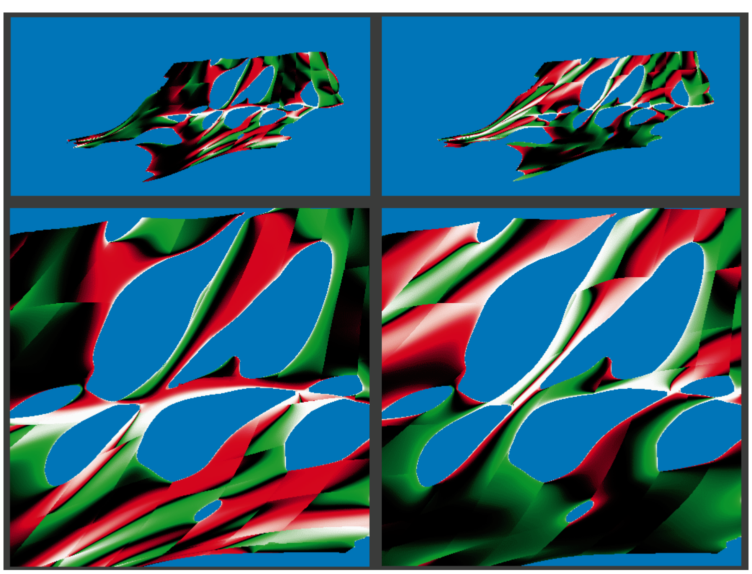
where *con* represents a visualization's contrast. κ_{norm} is color coded as shown in Figure 7. This way we obtain a color-coding map with the following properties: red for a positive value and green for a negative value. The higher the magnitude of the value, the lighter the color. In fact, a zero value gives black. If the value diverges to plus (minus) infinity, the red (green) color tends to white. The positive value *con* in Equation 23 can be considered



8 (a) Contour lines, (b) their thickness, (c) curvature, and (d) geodesic curvature.



9 (a) The two families of lines of curvature, (b) their geodesic curvatures, and (c) magnification of Figure 9b.



10 Geodesic curvature of the two families of asymptotic lines (top) and magnifications (bottom).

as the contrast of the visualization. Decreasing *con* leads to a darker picture but emphasizes the highlights. A visualization's contrast should be chosen interactively.

Using the color-coding map described above, critical points of the vector fields corresponding to the tangent curves can be detected as highlights in the visualization. If we render the surface using ray tracing, the choice of an appropriate number of surface points is easy—we simply use one surface point for every pixel point. For other rendering techniques we have to pick out sample points on the surface and interpolate between them.

Contour lines. Figure 8a shows the usual contour line visualization described in the section "Thickness of tangent curves." Figure 8b is the visualization of the contour line thickness. The negative value of the thickness was computed and color coded in the way described above for the curvature. This picture looks like a shaded image of the surface, though it is not. Figure 8c shows the curvature plot of the contour lines, and Figure 8d shows their geodesic curvature. All four pictures treat the same family of contour lines.

The geodesic curvature plot shows frequent changes of the curvature sign (for example, changes of red and green areas) in the right-hand part of the surface. This implies that the contour lines in this area have inflection points. This is hardly detectable from Figure 8a.

Because the visualization of the curvature and the geodesic curvature looks smooth, we cannot see the patch boundaries. This is an indicator for G^2 continuity of the surface.

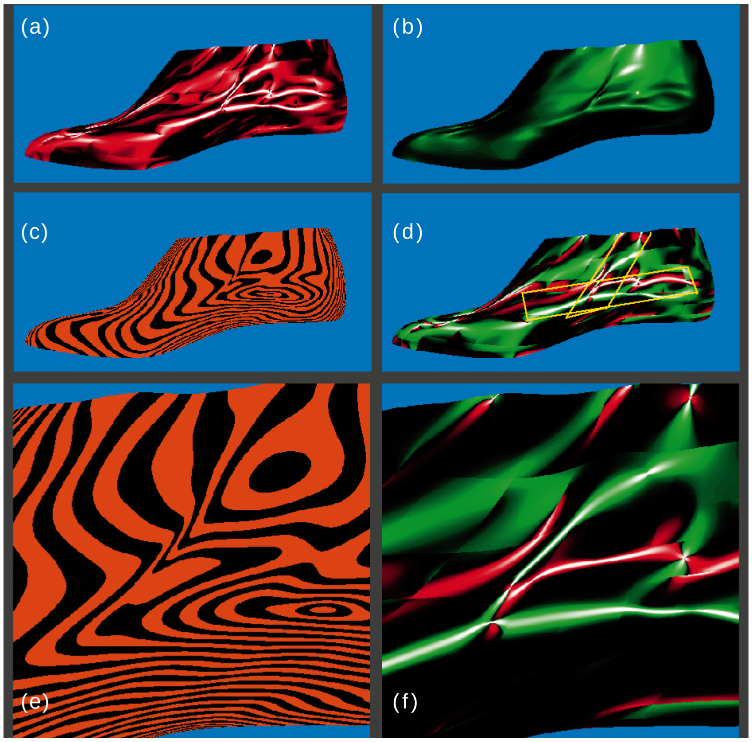
Lines of curvature. Figure 9a shows the numerical integration of the two families of the lines of curvature. Figure 9b shows their geodesic curvature. Figure 9c is a magnification of Figure 9b.

The curvature plots show two areas where the curvatures appear to be high and frequently changing—these areas are surrounded by a yellow line in Figure 9b. These are the areas where a redesign of the surface is necessary.

The curvature plots in Figure 9

show discontinuities at the patch boundaries. This shows that the surface is not G^3 continuous at the patch boundaries. The umbilical points of the surface can be detected as highlights in the curvature plots.

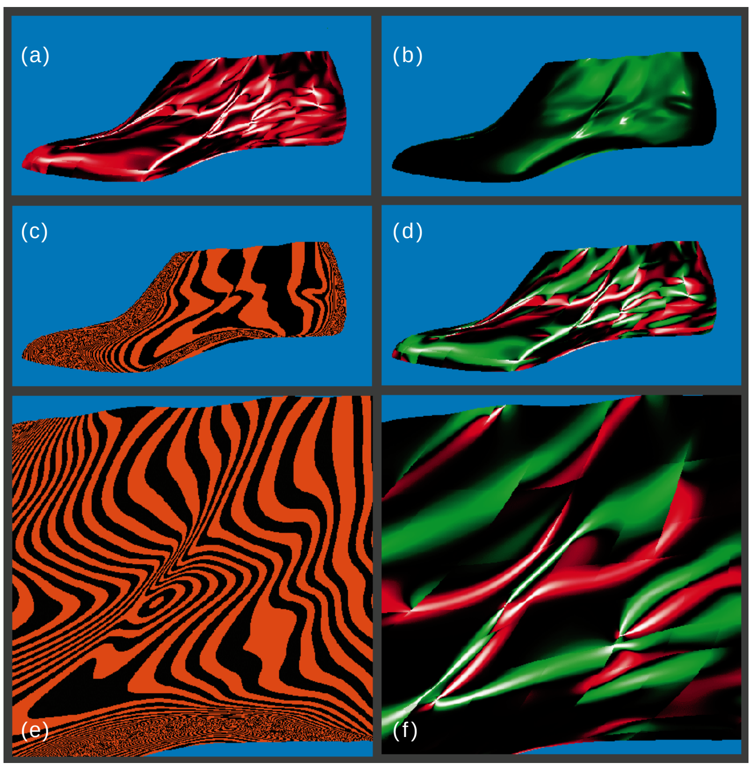
Asymptotic lines. The top of Figure 10 shows the geodesic curvature of the two families of asymptotic lines, and the bottom shows the magnifications. Since the asymptotic lines exist only for surface areas with a nonpositive Gaussian curvature, the areas with a positive Gaussian curvature are set to the background color. In our particular example, the curvature plot of asymptotic lines hardly provides new information in comparison to the curvature plot of the lines of curvature. This is because wide areas of the surface have a positive Gaussian curvature and therefore undefined asymptotic lines. However, discontinuities at the patch boundaries can be detected as well—the surface is not G^3 continuous.



11 (a) Curvature of isophotes, (b) their thickness, (c) the isophotes, (d) geodesic curvature, (e) magnification of Figure 11c, and (f) magnification of Figure 11d.

Isophotes. Figure 11c shows the visualization of a family of isophotes on the surface, and Figure 11b shows the color coding of their thickness. This picture clearly shows wrinkles, that is, strip-shaped areas where bright and dark colors change rapidly. These are the areas where the isophote thickness detects the necessity of a redesign. In comparison to the thickness of contour lines, the thickness of isophotes show more of these critical areas. This happens because the isophote thickness contains second order information about the surface, whereas the contour line thickness contains only first order information.

Figure 11a shows the curvature of the isophotes, while Figure 11d shows their geodesic curvature. Figures 11e and 11f show the magnifications of Figures 11c and 11d, respectively. Also, the curvature plots show areas of high and rapidly changing curvatures (surrounded by a yellow line). The curvature discontinuities at the patch boundaries indicate the not-G^3-property of the surface.



12 (a) Curvature of reflection lines, (b) their thickness, (c) reflection lines, (d) geodesic curvature, (e) magnification of Figure 12c, and (f) magnification of Figure 12d.

Reflection lines. Figure 12c shows the usual visualization of reflection lines on the surface. In this picture we can see aliasing effects near the edges of the surface. These effects do not appear in the thickness and curvature plots because those plots contain only continuous color changes (except for the patch boundaries).

Figure 12b shows the thickness of the reflection lines. Here again we can detect areas where the thickness changes rapidly—approximately in the same areas as in the isophote thickness visualization.

Figures 12a and 12d show the curvature and the geodesic curvature of the reflection lines. Figures 12e and 12f are magnifications of Figures 12c and 12d. The critical areas detected in these curvature plots coincide approximately with the areas detected by the isophote curvature. Also, as in the isophote case, the curvature discontinuities at the patch boundaries detect the not-G^3-property.

Conclusions

In this article we have shown that the visualization of the curvature of characteristic curves on surfaces is a useful interrogation tool. Next we plan to extend this into a surface-fairing tool. To obtain better curvature plots, algorithms must be developed that automatically correct a surface's shape. This approach seems promising because the curvature of characteristic curves contains parametrization-independent third order surface information. ■

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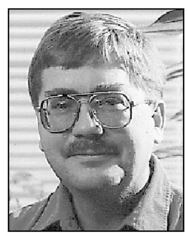

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