

Using Farin points for rational Bézier surfaces

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Abstract

Farin points (weight points) are a useful tool for handling the weights of rational Bézier curves. They describe the weights of the Bézier points uniquely and in a geometrically intuitive way.

The main problem for using Farin points for triangular or tensorproduct rational Bézier surfaces is the fact that they are not independent of each other and therefore overdefine the weights. To overcome this problem we present two approaches: moving a Farin point, we adjust the adjacent Farin points automatically in such a way that the system of all Farin points stays contradiction-free. In the other approach we present an appropriate subset of Farin points which are independent of each other and define the weights uniquely. Both approaches are presented for triangular and tensorproduct Bézier surfaces. They make Farin points useful for the design of rational Bézier surfaces. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Rational Bézier surfaces; Weights; Farin points (weight points)

1. Introduction

Farin points (weight points) are a useful design tool for rational Bézier curves. They provide an intuitive way of handling the weights of the Bézier points.

Given two adjacent Bézier points \mathbf{b}_i , \mathbf{b}_{i+1} with their assigned weights w_i , w_{i+1} on a Bézier polygon, the Farin point \mathbf{f}_i is defined as

$$\mathbf{f}_i = \frac{w_i \cdot \mathbf{b}_i + w_{i+1} \cdot \mathbf{b}_{i+1}}{w_i + w_{i+1}}. \quad (1)$$

The location of \mathbf{f}_i on the line through \mathbf{b}_i and \mathbf{b}_{i+1} determines the ratio of w_i and w_{i+1} uniquely and in an intuitive way. For positive weights, \mathbf{f}_i is located between \mathbf{b}_i and \mathbf{b}_{i+1} , and $\text{ratio}(\mathbf{b}_i, \mathbf{f}_i, \mathbf{b}_{i+1}) = w_{i+1}/w_i$. Intuitively: the larger w_i is relative to w_{i+1} , the closer \mathbf{f}_i moves toward \mathbf{b}_i . Fig. 1 gives an illustration. Given the Bézier polygon $\mathbf{b}_0, \dots, \mathbf{b}_n$,

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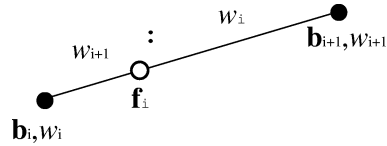


Fig. 1. Farin point f_i for the Bézier points b_i , b_{i+1} and their assigned weights w_i , w_{i+1} .

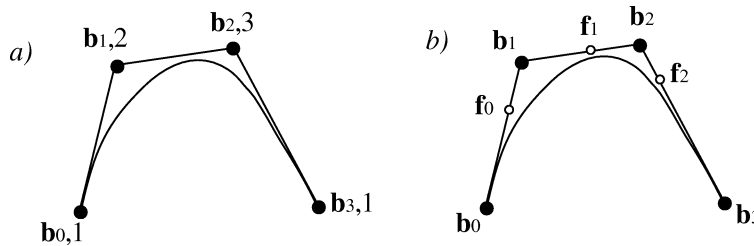


Fig. 2. Rational Bézier curve of degree 3 described by (a) Bézier points and assigned weights, (b) Bézier points and Farin points.

the assigned weights w_0, \dots, w_n can be described by the Farin points f_0, \dots, f_{n-1} . Fig. 2 shows an example for $n = 3$.

Farin points for rational Bézier curves have the following properties:

- Uniqueness: the Farin points f_0, \dots, f_{n-1} describe the weights w_0, \dots, w_n uniquely except for a common factor. This common factor has no influence on the curve shape.
- Independence: each of the Farin points f_0, \dots, f_{n-1} can be moved freely on the lines of the Bézier polygon. The adjacent Farin points are not affected by moving f_i .
- Intuitivity: instead of increasing or decreasing the weights a designer moves Farin points on the Bézier polygon. He or she may find this more intuitive.
- Extended convex hull: for positive weights, the rational Bézier curve lies not only in the convex hull of b_0, \dots, b_n , but also in the convex hull of $b_0, f_0, \dots, f_{n-1}, b_n$. Fig. 3 illustrates for $n = 3$.

A comprehensive introduction of rational Bézier curves and Farin points (weight points) can be found in (Farin, 1995).

This paper deals with the extension of Farin points to rational Bézier surfaces. A straightforward extension to surfaces creates problems because the independence property gets lost. As we will see in the next sections, Farin points for Bézier surfaces (both triangular and tensorproduct) overdefine the weights and have therefore dependencies to each other. In other words: moving a particular Farin point of a Bézier surface may cause contradictions in the system of all Farin points. A Farin point is no longer freely movable. An important design feature gets lost.

Section 2 shows the solutions for this problem for triangular Bézier surfaces in order to keep Farin points being a useful design tool. Section 3 does the same for tensorproduct surfaces.

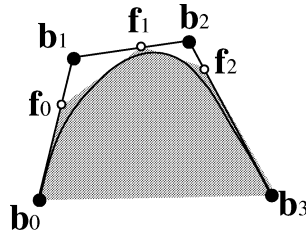


Fig. 3. Extended convex hull property for rational Bézier curves: the curve lies completely in the convex hull of $\mathbf{b}_0, \mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{b}_3$.

Notation. For describing basic geometric constructions in Euclidean space we use the following pseudo-code:

- $\mathbf{L} := \text{lin}(\mathbf{a}, \mathbf{b})$: let \mathbf{L} be the line through the points \mathbf{a} and \mathbf{b} .
- $\mathbf{L} := \text{par}(\mathbf{L}_0, \mathbf{a})$: let \mathbf{L} be the line parallel to the line \mathbf{L}_0 through \mathbf{a} .
- $\mathbf{a} := \text{int}(\mathbf{L}_1, \mathbf{L}_2)$: let \mathbf{a} be the intersection point of the lines \mathbf{L}_1 and \mathbf{L}_2 .

Furthermore we need the concept of cross ratio for four collinear points. We use the definition

$$\text{cr}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \frac{\text{ratio}(\mathbf{a}, \mathbf{b}, \mathbf{d})}{\text{ratio}(\mathbf{a}, \mathbf{c}, \mathbf{d})}. \tag{2}$$

2. Farin points for Bézier triangles

For triangular Bézier point schemes we have to find a way to describe the weight ratios of the three weights of a subtriangle.

For three Bézier points $\mathbf{b}_{i,j,k}, \mathbf{b}_{i+1,j,k-1}, \mathbf{b}_{i,j+1,k-1}$ with the assigned weights $w_{i,j,k}, w_{i+1,j,k-1}, w_{i,j+1,k-1}$, we define the Farin point as barycentric combination of the three Bézier points:

$$\mathbf{f}_{i,j,k} = \frac{w_{i,j,k} \cdot \mathbf{b}_{i,j,k} + w_{i+1,j,k-1} \cdot \mathbf{b}_{i+1,j,k-1} + w_{i,j+1,k-1} \cdot \mathbf{b}_{i,j+1,k-1}}{w_{i,j,k} + w_{i+1,j,k-1} + w_{i,j+1,k-1}}. \tag{3}$$

For positive weights, $\mathbf{f}_{i,j,k}$ is inside the triangle $\mathbf{b}_{i,j,k}, \mathbf{b}_{i+1,j,k-1}, \mathbf{b}_{i,j+1,k-1}$. Obviously, the location of $\mathbf{f}_{i,j,k}$ in the triangle determines the ratio of the three weights uniquely. Furthermore, we define the Farin points $\mathbf{p}_{i,j,k}, \mathbf{q}_{i,j,k}, \mathbf{r}_{i,j,k}$ on the edges of the triangle as determining the ratios of each two of the Bézier points:

$$\mathbf{p}_{i,j,k} = \frac{w_{i,j,k} \cdot \mathbf{b}_{i,j,k} + w_{i+1,j,k-1} \cdot \mathbf{b}_{i+1,j,k-1}}{w_{i,j,k} + w_{i+1,j,k-1}},$$

$$\mathbf{q}_{i,j,k} = \frac{w_{i+1,j,k-1} \cdot \mathbf{b}_{i+1,j,k-1} + w_{i,j+1,k-1} \cdot \mathbf{b}_{i,j+1,k-1}}{w_{i+1,j,k-1} + w_{i,j+1,k-1}},$$

$$\mathbf{r}_{i,j,k} = \frac{w_{i,j,k} \cdot \mathbf{b}_{i,j,k} + w_{i,j+1,k-1} \cdot \mathbf{b}_{i,j+1,k-1}}{w_{i,j,k} + w_{i,j+1,k-1}}.$$

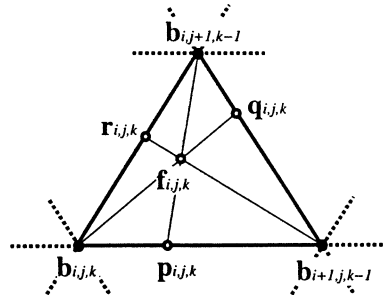


Fig. 4. Farin points $f_{i,j,k}$, $p_{i,j,k}$, $q_{i,j,k}$, $r_{i,j,k}$ for the Bézier triangle $b_{i,j,k}$, $b_{i+1,j,k-1}$, $b_{i,j+1,k-1}$.

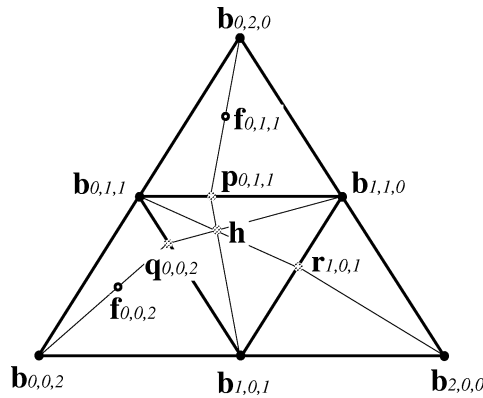


Fig. 5. Bézier triangle of order 2. Given the Farin points $f_{0,0,2}$, $f_{0,1,1}$, the Farin point $f_{1,0,1}$ is not freely movable. In fact $f_{1,0,1}$ must lie on $\text{lin}(r_{1,0,1}, b_{2,0,0})$.

The geometric correlation between $f_{i,j,k}$, $p_{i,j,k}$, $q_{i,j,k}$, $r_{i,j,k}$ is shown in Fig. 4.

Now we show that Farin points on Bézier triangles are not independent anymore. Consider a Bézier triangle of order 2, as shown in Fig. 5.

Suppose we know the Farin points $f_{0,0,2}$ and $f_{0,1,1}$. Then the following constructions give $r_{1,0,1}$:

$$q_{0,0,2} := \text{int}(\text{lin}(b_{1,0,1}, b_{0,1,1}), \text{lin}(b_{0,0,2}, f_{0,0,2})),$$

$$p_{0,1,1} := \text{int}(\text{lin}(b_{0,1,1}, b_{1,1,0}), \text{lin}(b_{0,2,0}, f_{0,1,1})),$$

$$h := \text{int}(\text{lin}(p_{0,1,1}, b_{1,0,1}), \text{lin}(q_{0,0,2}, b_{1,1,0})),$$

$$r_{1,0,1} := \text{int}(\text{lin}(b_{1,0,1}, b_{1,1,0}), \text{lin}(b_{0,1,1}, h)).$$

The Farin point $f_{1,0,1}$ must lie on $\text{lin}(r_{1,0,1}, b_{2,0,0})$. It is not freely movable any more. If $f_{1,0,1}$ is not on $\text{lin}(r_{1,0,1}, b_{2,0,0})$, the system of the Farin points $f_{0,0,2}$, $f_{0,1,1}$, $f_{1,0,1}$ is not contradiction-free. $f_{0,0,2}$, $f_{0,1,1}$ and $f_{1,0,1}$ are not independent of each other.

To overcome this problem (and therefore make Farin points on Bézier triangles usable as a design tool) there are two approaches:

- (a) Allow the user to move every Farin point and adjust the adjacent Farin points simultaneously so that the system of all Farin points stays contradiction-free.
- (b) Offer the user not all Farin points to move but only a certain number. These Farin points should be independent of each other and describe all weights of the Bézier points uniquely (except for a common factor).

Approach (a) is treated in Section 2.1, Section 2.2 shows solutions for approach (b).

2.1. Adjusting adjacent Farin points

Given a triangular Bézier point scheme, we consider all Farin points $f_{i,j,k}$ of “upright” Bézier subtriangles. As we know from the example in Fig. 5, these Farin points are not independent of each other.

Fig. 6 shows a part of triangular Bézier point scheme. Suppose the system of all Farin points $f_{i,j,k}$ is contradiction-free. After moving $f_{0,0,0}$ to $\widetilde{f}_{0,0,0}$, the system of Farin points is generally not contradiction-free any more. In order to preserve the freedom of contradictions, we adjust the adjacent Farin points

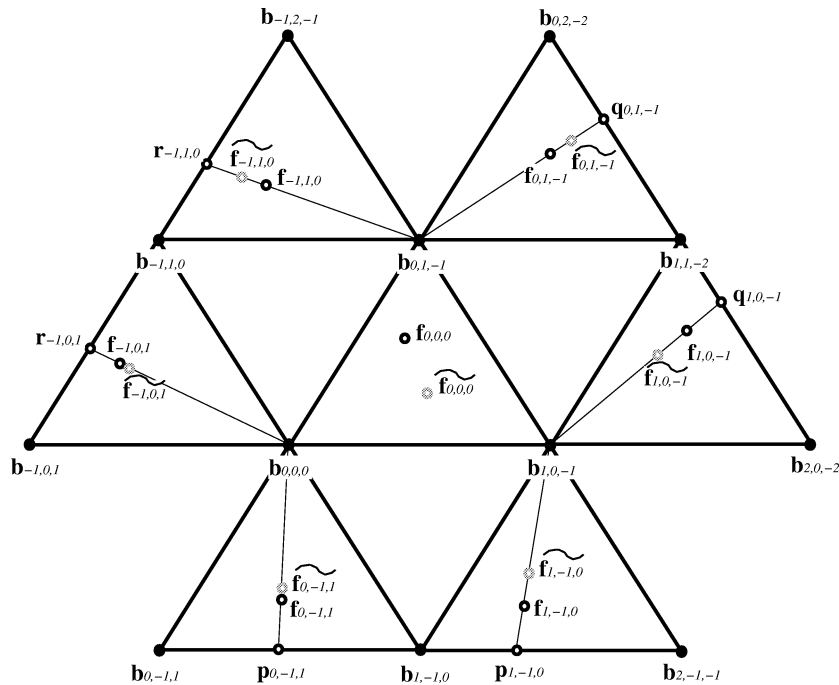


Fig. 6. Moving the Farin point $f_{0,0,0}$ to $\widetilde{f}_{0,0,0}$. In order to preserve the freedom of contradictions, the adjacent Farin points $f_{0,1,-1}$, $f_{-1,1,0}$, $f_{-1,0,1}$, $f_{0,-1,1}$, $f_{1,-1,0}$ have to be adjusted to $\widetilde{f}_{0,1,-1}$, $\widetilde{f}_{-1,1,0}$, $\widetilde{f}_{-1,0,1}$, $\widetilde{f}_{0,-1,1}$, $\widetilde{f}_{1,-1,0}$.

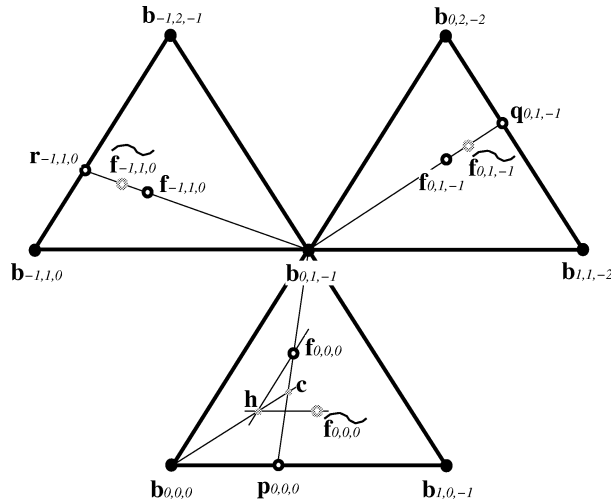


Fig. 7. Moving the Farin point $\widetilde{f_{0,0,0}}$ to $\widetilde{f_{0,0,0}}$ and adjusting the adjacent Farin points. Constructing the auxiliary point \widetilde{c} , we obtain: $\text{cr}(\mathbf{b}_{0,1,-1}, \widetilde{f_{-1,1,0}}, \widetilde{f_{-1,1,0}}, \mathbf{r}_{-1,1,0}) = \text{cr}(\mathbf{b}_{0,1,-1}, \widetilde{f_{0,1,-1}}, \widetilde{f_{0,1,-1}}, \mathbf{q}_{0,1,-1}) = \text{cr}(\mathbf{b}_{0,1,-1}, \widetilde{f_{0,0,0}}, \mathbf{c}, \mathbf{p}_{0,0,0})$.

$$\mathbf{f}_{0,-1,1}, \mathbf{f}_{1,-1,0}, \mathbf{f}_{1,0,-1}, \mathbf{f}_{0,1,-1}, \mathbf{f}_{-1,0,1}, \mathbf{f}_{-1,1,0}$$

to the new points

$$\widetilde{f_{0,-1,1}}, \widetilde{f_{1,-1,0}}, \widetilde{f_{1,0,-1}}, \widetilde{f_{0,1,-1}}, \widetilde{f_{-1,0,1}}, \widetilde{f_{-1,1,0}}.$$

We know that $\widetilde{f_{0,1,-1}}$ lies on $\text{lin}(\mathbf{b}_{0,1,-1}, \mathbf{f}_{0,1,-1})$. The similar statements for the other Farin points can be seen in Fig. 6. Furthermore, it can be shown that:

$$\text{cr}(\mathbf{b}_{0,1,-1}, \mathbf{f}_{0,1,-1}, \widetilde{f_{0,1,-1}}, \mathbf{q}_{0,1,-1}) = \text{cr}(\mathbf{b}_{0,1,-1}, \mathbf{f}_{-1,1,0}, \widetilde{f_{-1,1,0}}, \mathbf{r}_{-1,1,0}),$$

$$\text{cr}(\mathbf{b}_{0,0,0}, \mathbf{f}_{-1,0,1}, \widetilde{f_{-1,0,1}}, \mathbf{r}_{-1,0,1}) = \text{cr}(\mathbf{b}_{0,0,0}, \mathbf{f}_{0,-1,1}, \widetilde{f_{0,-1,1}}, \mathbf{p}_{0,-1,1}),$$

$$\text{cr}(\mathbf{b}_{1,0,-1}, \mathbf{f}_{1,-1,0}, \widetilde{f_{1,-1,0}}, \mathbf{p}_{1,-1,0}) = \text{cr}(\mathbf{b}_{1,0,-1}, \mathbf{f}_{1,0,-1}, \widetilde{f_{1,0,-1}}, \mathbf{q}_{1,0,-1}).$$

All we have to do now is to determine these cross ratios. For doing this, consider Fig. 7. We constructed the auxiliary point \widetilde{c} in the following way:

$$\mathbf{h} := \text{int}(\text{par}(\text{lin}(\mathbf{b}_{0,0,0}, \mathbf{b}_{1,0,-1}), \widetilde{f_{0,0,0}}), \text{par}(\text{lin}(\mathbf{b}_{0,0,0}, \mathbf{b}_{0,1,-1}), \mathbf{f}_{0,0,0})),$$

$$\mathbf{c} := \text{int}(\text{lin}(\mathbf{b}_{0,0,0}, \mathbf{h}), \text{lin}(\mathbf{b}_{0,1,-1}, \mathbf{f}_{0,0,0})).$$

Then it can be shown that

$$\text{cr}(\mathbf{b}_{0,1,-1}, \mathbf{f}_{0,0,0}, \mathbf{c}, \mathbf{p}_{0,0,0}) = \text{cr}(\mathbf{b}_{0,1,-1}, \mathbf{f}_{0,1,-1}, \widetilde{f_{0,1,-1}}, \mathbf{q}_{0,1,-1})$$

$$= \text{cr}(\mathbf{b}_{0,1,-1}, \mathbf{f}_{-1,1,0}, \widetilde{f_{-1,1,0}}, \mathbf{r}_{-1,1,0}).$$

Therefore, the new adjusted Farin points $\widetilde{f_{0,1,-1}}$ and $\widetilde{f_{-1,1,0}}$ can be geometrically constructed. Similar constructions apply for the other adjacent Farin points.

2.2. Independent Farin points for Bézier triangles

In this subsection we want to establish a system of independent Farin points for Bézier triangles which describes the weights of the Bézier points uniquely (except for a common factor). The first solution of this was published in (Albrecht, 1995). Here the problem was reduced to the curve problem by using not the Farin points $\mathbf{f}_{i,j,k}$ in the subtriangles but the points $\mathbf{p}_{i,j,k}$, $\mathbf{q}_{i,j,k}$, $\mathbf{r}_{i,j,k}$ on the edges of the subtriangles (see Fig. 4). A system of these points—each of them movable on a line segment—gave the solution. This solution was obtained by searching a spanning tree for the control net which is interpreted as a graph.

In this subsection we use basic ideas of (Albrecht, 1995) but present a system of independent Farin points $\mathbf{f}_{i,j,k}$, i.e., these Farin points are freely movable inside a whole triangle. Therefore we need fewer Farin points than in (Albrecht, 1995).

First we have to establish that such a system of independent Farin points exists at all. In doing this we keep in mind that a Farin point $\mathbf{f}_{i,j,k}$ is movable in 2D and therefore its location covers 2 degrees of freedom.

The number of degrees of freedom we have to cover in a triangular Bézier point scheme of the order n is:

$$df(n) = \frac{(n + 1)(n + 2)}{2} - 1. \tag{4}$$

This means that $df(n)$ is the number of the Bézier points minus 1. We have to fix the weights of all Bézier points except for a common factor. One weight can be chosen randomly, then the other weights are fixed.

Table 1 shows the number $df(n)$ of degrees of freedom for small n .

If $df(n)$ is even, we can find a system of $df(n)/2$ independent Farin points. For $df(n)$ is odd, there are two strategies:

- (α) Find $(df(n) - 1)/2$ independent Farin points and determine the weight of one corner point of the Bézier triangle explicitly.
- (β) Find $(df(n) - 3)/2$ independent Farin points and determine the weights of all three corner points of the Bézier triangle explicitly. This strategy has the advantage of preserving symmetry in the triangular scheme.

Fig. 8 shows the solution for strategy (α) for small n . In this figure, all subtriangles for which a Farin point is used are marked. For $n = 2, 3, 6$ we have $df(n)$ odd. For the special treatment of the corners lower left we introduced Farin points on line segments which relate the weight of the corner to the average of the weights of the other two Bézier points in the subtriangle.

Table 1
Number $df(n)$ of degrees of freedom to be covered by Farin points in triangular Bézier point schemes of the order n

n	1	2	3	4	5	6	7	...
$df(n)$	2	5	9	14	20	27	35	...
	even	odd	odd	even	even	odd	odd	...

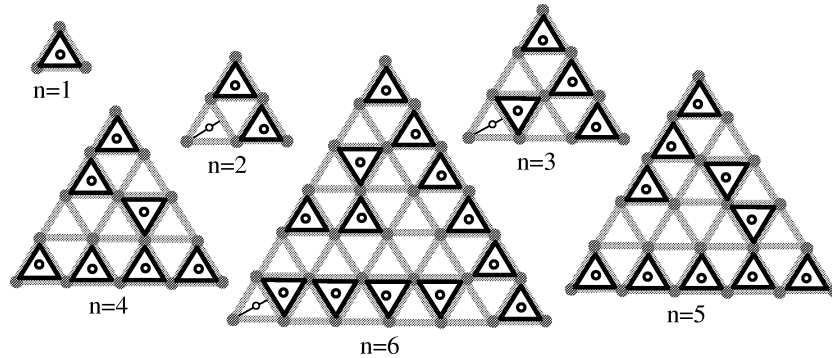


Fig. 8. Schemes of independent Farin points for Bézier triangles of degree 1–6, strategy (α). Marked are all subtriangles in which the Farin point is considered. For $n = 2, 3, 6$ the weights of one corner Bézier point was fixed by introducing a Farin point on a line segment.

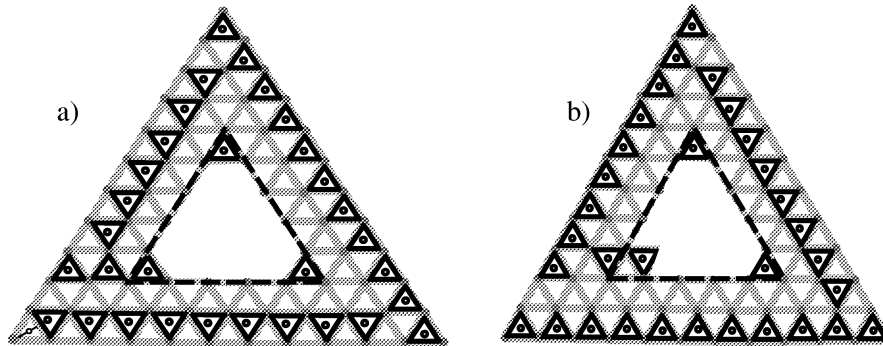


Fig. 9. Obtaining the system of independent Farin points for the degree $n + 6$ from a given solution for degree n , using strategy (α): (a) for $df(n)$ even, (b) for $df(n)$ odd.

Fig. 9 shows how to get the solution for degree $n + 6$ from a given solution for degree n . Figs. 8 and 9 give the solution for strategy (α) for any degree n by induction. Figs. 10 and 11 show the solution for strategy (β) by induction.

Fig. 12 shows an example: the system of independent Farin points for a triangular Bézier point scheme of order 13.

2.2.1. Remarks

- (1) The schemes introduced in this subsection use Farin points both on “upright” and “upside down” subtriangles. This might be unusual because the triangular de Casteljau algorithm works only on “upright” subtriangles. But the additional consideration of “upside down” subtriangles is necessary for finding systems of independent Farin points.
- (2) The systems of Farin points introduced in this subsection have the independence property but *not* the local control property. This means that moving one of the

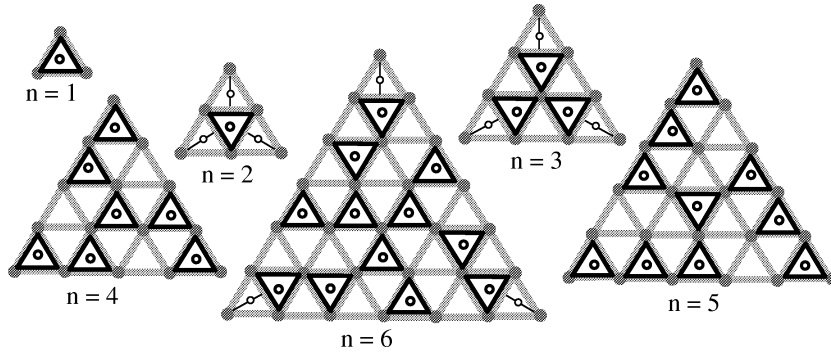


Fig. 10. Schemes of independent Farin points for Bézier triangles of degree 1–6, strategy (β).

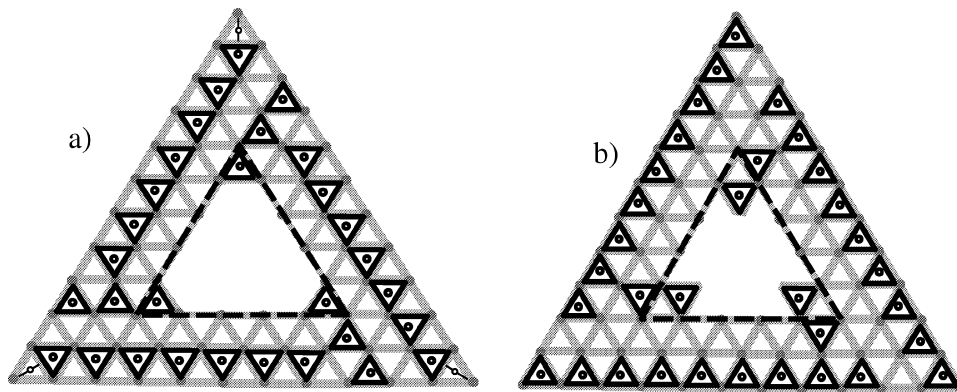


Fig. 11. Obtaining the system of independent Farin points for the degree $n + 6$ from a given solution for degree n , using strategy (β): (a) for $df(n)$ even, (b) for $df(n)$ odd.

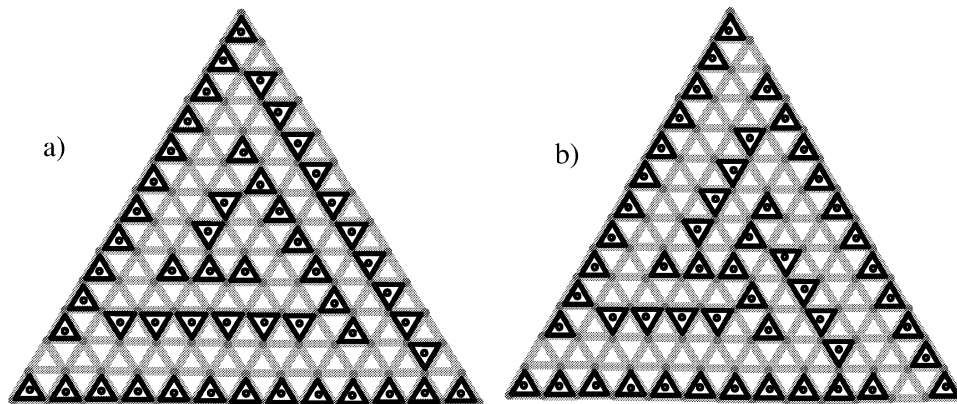


Fig. 12. System of independent Farin points for $n = 13$, using the strategy (α) and (β).

marked Farin points might change any other unmarked Farin point. If the local control property of Farin points is important we have to apply the automatic adjusting introduced in Section 2.1. A system with both the independence property and the local control property seems not to exist.

- (3) All schemes and constructions introduced in Section 2 work both in the domain of the Bézier triangles and in the Bézier point scheme in 3D. This is because the Bézier point subtriangles in 3D can be considered as affine maps of the subtriangles in the domain, and all constructions are affine invariant.

3. Farin points for tensorproduct Bézier surfaces

To handle the weights of a rectangular Bézier point scheme by Farin points, we first have to solve the problem for a subquadrilateral. We look for a Farin point solution for a subquadrilateral both in the domain of the surface and in 3D.

For the domain case, the Bézier points $\mathbf{b}_{i,j}$, $\mathbf{b}_{i+1,j}$, $\mathbf{b}_{i+1,j+1}$, $\mathbf{b}_{i,j+1}$ form a rectangle, see Fig. 13(a). Then we can define the points on the edges of the rectangle which define the ratios of each two of the weights:

$$\mathbf{p}_{i,j} = \frac{w_{i,j} \cdot \mathbf{b}_{i,j} + w_{i+1,j} \cdot \mathbf{b}_{i+1,j}}{w_{i,j} + w_{i+1,j}},$$

$$\mathbf{p}_{i,j+1} = \frac{w_{i,j+1} \cdot \mathbf{b}_{i,j+1} + w_{i+1,j+1} \cdot \mathbf{b}_{i+1,j+1}}{w_{i,j+1} + w_{i+1,j+1}},$$

$$\mathbf{q}_{i,j} = \frac{w_{i,j} \cdot \mathbf{b}_{i,j} + w_{i,j+1} \cdot \mathbf{b}_{i,j+1}}{w_{i,j} + w_{i,j+1}},$$

$$\mathbf{q}_{i+1,j} = \frac{w_{i+1,j} \cdot \mathbf{b}_{i+1,j} + w_{i+1,j+1} \cdot \mathbf{b}_{i+1,j+1}}{w_{i+1,j} + w_{i+1,j+1}}.$$

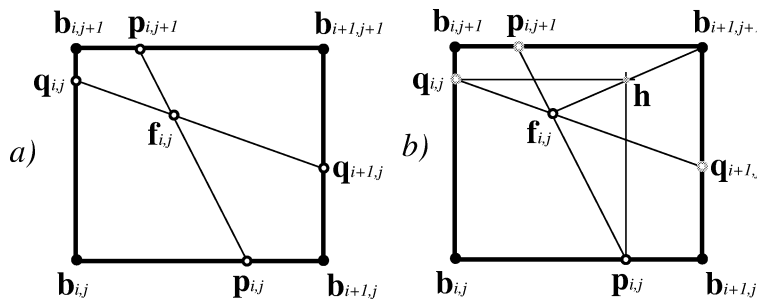


Fig. 13. (a) Defining the points $\mathbf{p}_{i,j}$, $\mathbf{p}_{i,j+1}$, $\mathbf{q}_{i,j}$, $\mathbf{q}_{i+1,j}$ and the Farin point $\mathbf{f}_{i,j}$ for the Bézier points $\mathbf{b}_{i,j}$, $\mathbf{b}_{i+1,j}$, $\mathbf{b}_{i+1,j+1}$, $\mathbf{b}_{i,j+1}$ assigned with the weights $w_{i,j}$, $w_{i+1,j}$, $w_{i+1,j+1}$, $w_{i,j+1}$. (b) Constructing $\mathbf{q}_{i,j}$, $\mathbf{q}_{i+1,j}$ and $\mathbf{p}_{i,j+1}$ from the given points $\mathbf{f}_{i,j}$ and $\mathbf{p}_{i,j}$ geometrically.

Furthermore we define the Farin point

$$\begin{aligned} \mathbf{f}_{i,j} &= \text{int}(\text{lin}(\mathbf{p}_{i,j}, \mathbf{p}_{i,j+1}), \text{lin}(\mathbf{q}_{i,j}, \mathbf{q}_{i+1,j})) \\ &= \frac{w_{i,j} \cdot \mathbf{b}_{i,j} + w_{i+1,j} \cdot \mathbf{b}_{i+1,j} + w_{i+1,j+1} \cdot \mathbf{b}_{i+1,j+1} + w_{i,j+1} \cdot \mathbf{b}_{i,j+1}}{w_{i,j} + w_{i+1,j} + w_{i+1,j+1} + w_{i,j+1}}. \end{aligned}$$

See Fig. 13(a) for an illustration. The Farin point $\mathbf{f}_{i,j}$ has the intuitivity property: increasing the weight of one Bézier point leads to moving $\mathbf{f}_{i,j}$ towards this Bézier point. Unfortunately, $\mathbf{f}_{i,j}$ is not sufficient to define uniquely the weight ratios in a subquadrilateral. In fact $\mathbf{f}_{i,j}$ is freely movable in 2D and therefore covers two degrees of freedom. What we have to determine are three degrees of freedom in a quadrilateral (the weight of one Bézier point can be chosen randomly, then the weights of the other three Bézier points have to be fixed).

Suppose the points $\mathbf{f}_{i,j}$ and $\mathbf{p}_{i,j}$ are given. Then the remaining points $\mathbf{p}_{i,j+1}$, $\mathbf{q}_{i,j}$ and $\mathbf{q}_{i+1,j}$ can be geometrically constructed in the following way:

$$\begin{aligned} \mathbf{p}_{i,j+1} &:= \text{int}(\text{lin}(\mathbf{p}_{i,j}, \mathbf{f}_{i,j}), \text{lin}(\mathbf{b}_{i,j+1}, \mathbf{b}_{i+1,j+1})), \\ \mathbf{h} &:= \text{int}(\text{lin}(\mathbf{f}_{i,j}, \mathbf{b}_{i+1,j+1}), \text{par}(\text{lin}(\mathbf{b}_{i+1,j}, \mathbf{b}_{i+1,j+1}), \text{lin}(\mathbf{p}_{i,j}, \mathbf{b}_{i,j}))), \\ \mathbf{q}_{i,j} &:= \text{int}(\text{lin}(\mathbf{b}_{i,j}, \mathbf{b}_{i,j+1}), \text{par}(\text{lin}(\mathbf{b}_{i,j+1}, \mathbf{b}_{i+1,j+1}), \mathbf{h})), \\ \mathbf{q}_{i+1,j} &:= \text{int}(\text{lin}(\mathbf{q}_{i,j}, \mathbf{f}_{i,j}), \text{lin}(\mathbf{b}_{i+1,j}, \mathbf{b}_{i+1,j+1})). \end{aligned}$$

See Fig. 13(b) for an illustration. For reasons of simplification, we consider from now on the Bézier point subquadrilateral $\mathbf{b}_{0,0}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1}, \mathbf{b}_{0,1}$. The points $\mathbf{p}_{0,0}, \mathbf{q}_{1,0}, \mathbf{p}_{0,1}, \mathbf{q}_{0,0}$ are not independent of each other. In fact, the location of three of these points determines the remaining one. To find a construction for this fact we consider a property illustrated in Fig. 14(b): the three lines $\text{lin}(\mathbf{b}_{0,0}, \mathbf{b}_{1,1})$, $\text{lin}(\mathbf{p}_{0,0}, \mathbf{q}_{1,0})$ and $\text{lin}(\mathbf{q}_{0,0}, \mathbf{p}_{0,1})$ either intersect at one point or are all parallel. As shown in Fig. 14(a), the same property is true in 3D.

Given the Farin point $\mathbf{f}_{0,0}$ in the rectangular domain quadrilateral, we seek a way to determine all weight ratios. One way of doing this is fixing one of the weight points on the edges, as shown in Fig. 13(b). This approach is not symmetric because we have to make

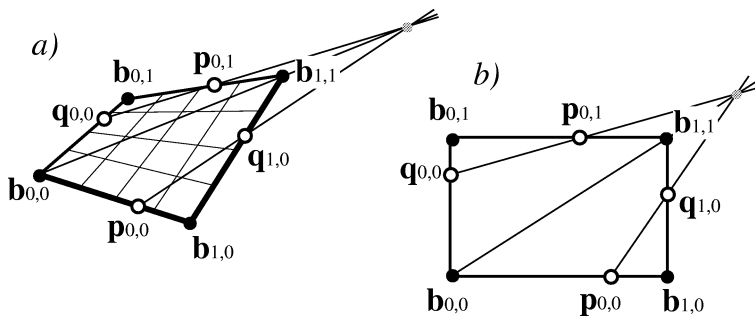


Fig. 14. Weight points on the edges of a subquadrilateral: the lines $\text{lin}(\mathbf{b}_{0,0}, \mathbf{b}_{1,1})$, $\text{lin}(\mathbf{p}_{0,0}, \mathbf{q}_{1,0})$ and $\text{lin}(\mathbf{q}_{0,0}, \mathbf{p}_{0,1})$ either intersect at one point or are parallel. This is true both in the domain case (b) and in the 3D case (a).

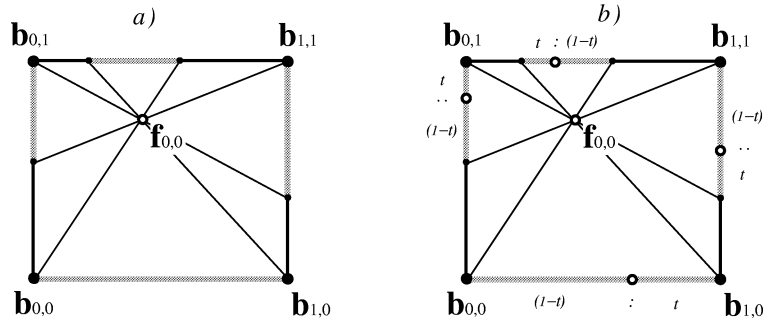


Fig. 15. (a) Constructing permitted areas (grey marked) on the edges of the rectangle by intersecting the lines $\text{lin}(\mathbf{b}_{0,0}, \mathbf{f}_{0,0})$, $\text{lin}(\mathbf{b}_{1,0}, \mathbf{f}_{0,0})$, $\text{lin}(\mathbf{b}_{1,1}, \mathbf{f}_{0,0})$ and $\text{lin}(\mathbf{b}_{0,1}, \mathbf{f}_{0,0})$ with the rectangle. (b) Dividing the permitted areas in the ratio $t/(1-t)$. We obtain the weight points on the edges of the quadrilateral.

the choice which point on the edges to fix. Now we want to introduce a symmetric way of defining the weight ratios using the Farin point $\mathbf{f}_{0,0}$.

Consider Fig. 15(a). We assume that the weights $w_{0,0}$, $w_{1,0}$, $w_{1,1}$, $w_{0,1}$ which are assigned to the Bézier points $\mathbf{b}_{0,0}$, $\mathbf{b}_{0,1}$, $\mathbf{b}_{1,1}$, $\mathbf{b}_{1,0}$ are all positive. This means that the point $\mathbf{p}_{0,0}$ lies inbetween $\mathbf{b}_{0,0}$ and $\mathbf{b}_{1,0}$, similar for the other three weight points on the edges. Now we seek all possible locations for $\mathbf{p}_{0,1}$ between $\mathbf{b}_{0,1}$ and $\mathbf{b}_{1,1}$, so that $\mathbf{p}_{0,0}$ is between $\mathbf{b}_{0,0}$ and $\mathbf{b}_{1,0}$.

Obviously, this is the grey marked area on the line $\mathbf{b}_{0,1}$, $\mathbf{b}_{1,1}$. We call this area the *permitted area*.

The permitted area of an edge of the rectangle is the area where the weight point can lie so that the weight point of the opposite edge is inbetween the corner points. In Fig. 15(a), the permitted area of the edge $\mathbf{b}_{0,0}$, $\mathbf{b}_{1,0}$ is the whole line segment $\mathbf{b}_{0,0}$, $\mathbf{b}_{1,0}$: no matter where $\mathbf{p}_{0,0}$ is located, the opposite point $\mathbf{p}_{0,1}$ will be in between $\mathbf{b}_{0,1}$, $\mathbf{b}_{1,1}$.

The location of the permitted areas on the edges depends on the location of $\mathbf{f}_{0,0}$. All permitted areas can be found by intersecting the four lines $\text{lin}(\mathbf{b}_{0,0}, \mathbf{f}_{0,0})$, $\text{lin}(\mathbf{b}_{1,0}, \mathbf{f}_{0,0})$, $\text{lin}(\mathbf{b}_{1,1}, \mathbf{f}_{0,0})$, $\text{lin}(\mathbf{b}_{0,1}, \mathbf{f}_{0,0})$ with the rectangle. See Fig. 15(a) for an illustration.

Now we divide the four permitted areas in the ratio $t/(1-t)$, as shown in Fig. 15(b) (with $0 < t < 1$). The resulting four points on the edges of the rectangle can be considered as the weight points on the edges. It can be shown that these four points have the property described in Fig. 14(b). This means that we have described the weight ratios in the quadrilateral completely by $\mathbf{f}_{0,0}$ and t . For finding a geometric meaning of t , we consider Fig. 15(b) again. The more t tends to 0, the closer the weight points tend to $\mathbf{b}_{0,1}$ or $\mathbf{b}_{1,0}$. The more t tends to 1, the closer the weight points tend to $\mathbf{b}_{0,0}$ or $\mathbf{b}_{1,1}$. This means that the parameter t is a measure which diagonal $\mathbf{b}_{0,0}$, $\mathbf{b}_{1,1}$ or $\mathbf{b}_{0,1}$, $\mathbf{b}_{1,0}$ is more emphasized. This gives reason for introducing *extended Farin points*.

An extended Farin point is a freely movable icon as shown in Fig. 16. The location of the lower hollow point is the location of the Farin point $\mathbf{f}_{0,0}$. The upper hollow point is freely movable on the horizontal line segment and fixes which of the diagonals $\mathbf{b}_{1,0}$, $\mathbf{b}_{0,1}$ or $\mathbf{b}_{0,0}$, $\mathbf{b}_{1,1}$ is more emphasized. The diagonals are symbolized by the nonhorizontal lines.

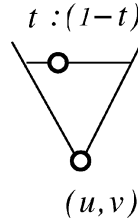


Fig. 16. Extended Farin point: freely movable icon in the rectangular domain. The location of the lower hollow point gives $\mathbf{f}_{0,0}$. The upper hollow point determines the parameter t .

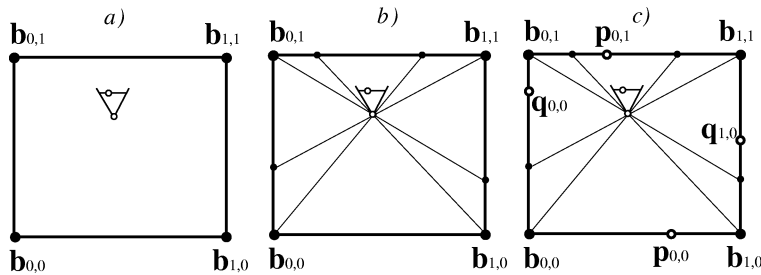


Fig. 17. Usage of extended Farin points. (a) Given is Bézier point rectangle $\mathbf{b}_{0,0}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1}, \mathbf{b}_{0,1}$ and the extended Farin point inside the rectangle. (b) Construct the permitted areas on the edges of the rectangle. (c) Divide the permitted areas in the same ratio as the upper hollow point divides the horizontal line segment of the extended Farin point. The resulting four weight points $\mathbf{p}_{0,0}, \mathbf{q}_{1,0}, \mathbf{p}_{0,1}, \mathbf{q}_{1,1}$ define the ratios of the weights of the four Bézier points uniquely.

An extended Farin point describes the weight ratios of the Bézier points in a subquadrilateral uniquely and symmetrically: no particular edge or corner of the rectangle has to be chosen. The conditions of independence and intuitivity are also fulfilled by extended Farin points. An example of the usage of an extended Farin point is given in Fig. 17.

Farin points in 3D

Up to here, this chapter has treated Farin points in the rectangular domain of a quadrilateral. Now we consider four Bézier points $\mathbf{b}_{0,0}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1}, \mathbf{b}_{0,1}$ as living in the 3D space of the surfaces. The straightforward approach here to determine the weight ratios is to consider the barycentric combination

$$\mathbf{f} = \frac{w_{0,0} \cdot \mathbf{b}_{0,0} + w_{1,0} \cdot \mathbf{b}_{1,0} + w_{1,1} \cdot \mathbf{b}_{1,1} + w_{0,1} \cdot \mathbf{b}_{0,1}}{w_{0,0} + w_{1,0} + w_{1,1} + w_{0,1}} \tag{5}$$

For positive weights, \mathbf{f} lies in the convex hull of $\mathbf{b}_{0,0}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1}, \mathbf{b}_{0,1}$. In general \mathbf{f} is movable in 3D and thus covers the three degrees of freedom we have to fix. Unfortunately, this approach fails completely if $\mathbf{b}_{0,0}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1}, \mathbf{b}_{0,1}$ are coplanar, and it fails numerically if $\mathbf{b}_{0,0}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1}, \mathbf{b}_{0,1}$ are almost coplanar. Since in practical applications the case of

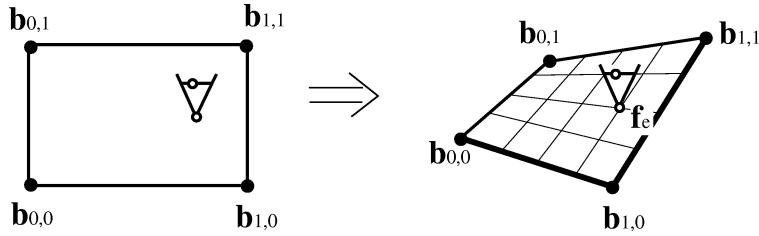


Fig. 18. The extended Farin point in the 2D domain is mapped onto the bilinear interpolant of $\mathbf{b}_{0,0}$, $\mathbf{b}_{1,0}$, $\mathbf{b}_{1,1}$, $\mathbf{b}_{0,1}$ in 3D.

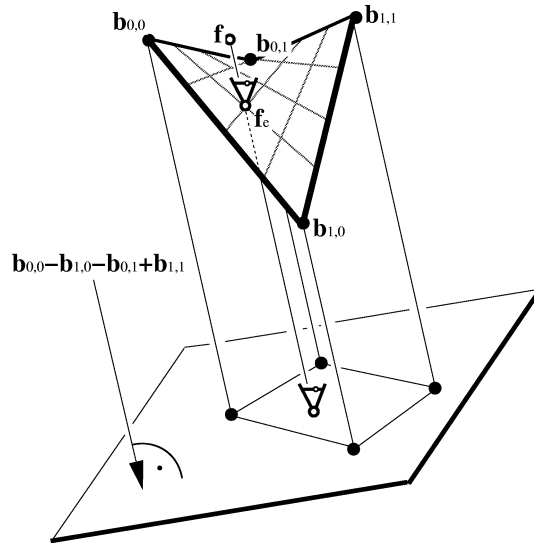


Fig. 19. Getting the weight ratios from an extended Farin point \mathbf{f}_e on the bilinear interpolation: apply a projection in the twist vector direction $\mathbf{b}_{0,0} - \mathbf{b}_{1,0} - \mathbf{b}_{0,1} + \mathbf{b}_{1,1}$. The projection of the four Bézier points gives a parallelogram inside which we can apply all constructions for extended Farin points in the 2D domain. Note that the projections of the location \mathbf{f}_e of the extended Farin point and the barycentric combination \mathbf{f} introduced in (5) are identical.

almost planar subquadrilaterals is common, we have to find another way to describe the weight ratios: we use the concept of the extended Farin points in 2D and map this onto the bilinear interpolant of $\mathbf{b}_{0,0}$, $\mathbf{b}_{1,0}$, $\mathbf{b}_{1,1}$, $\mathbf{b}_{0,1}$ in 3D. Fig. 18 gives an illustration.

The extended Farin point in 3D is freely movable on the bilinear interpolant. The location \mathbf{f}_e of the extended Farin point on the bilinear interpolant can be computed as

$$\mathbf{f}_e = \frac{hw_{0,0} \cdot \mathbf{b}_{0,0} + hw_{1,0} \cdot \mathbf{b}_{1,0} + hw_{1,1} \cdot \mathbf{b}_{1,1} + hw_{0,1} \cdot \mathbf{b}_{0,1}}{hw_{0,0} + hw_{1,0} + hw_{1,1} + hw_{0,1}} \quad (6)$$

with

$$\begin{aligned}
 hw_{0,0} &= (w_{0,0} + w_{1,0}) \cdot (w_{0,0} + w_{0,1}), & hw_{1,0} &= (w_{1,0} + w_{0,0}) \cdot (w_{1,0} + w_{1,1}), \\
 hw_{1,1} &= (w_{1,1} + w_{1,0}) \cdot (w_{1,1} + w_{0,1}), & hw_{0,1} &= (w_{0,1} + w_{0,0}) \cdot (w_{0,1} + w_{1,1}).
 \end{aligned}$$

In order to get the weight ratios out of \mathbf{f}_e , we apply a parallel projection in the direction of the twist vector $\mathbf{b}_{0,0} - \mathbf{b}_{1,0} - \mathbf{b}_{0,1} + \mathbf{b}_{1,1}$. Doing this the four Bézier points happen to form a parallelogram with the projection of \mathbf{f}_e inside it. In this parallelogram we can carry out all constructions introduced in this section for the domain case. See Fig. 19 for an illustration.

3.1. Adjusting adjacent Farin points in rectangular Bézier point schemes

After showing how to handle the weights in a subquadrilateral by using the concept of extended Farin points, we now treat the case of a whole rectangular Bézier point scheme. As in the triangular case there are two ways for keeping the system of all Farin points contradiction-free:

- (a) Allow every Farin point to be movable and adjust the adjacent Farin points.
- (b) Provide a system of Farin points which are independent of each other and describe the weight ratios uniquely.

This subsection treats approach (a); a solution for (b) is given in Section 3.2.

Consider Fig. 20 for a rectangular Farin point scheme in the domain. Suppose the weight ratios in the subquadrilateral $\mathbf{b}_{1,1}$, $\mathbf{b}_{2,1}$, $\mathbf{b}_{2,2}$, $\mathbf{b}_{1,2}$ are changed, for instance using the concepts of an extended Farin point. This means that the points $\mathbf{f}_{1,1}$, $\mathbf{p}_{1,1}$, $\mathbf{q}_{2,1}$, $\mathbf{p}_{1,2}$, $\mathbf{q}_{1,1}$ are moved to the new locations $\widetilde{\mathbf{f}}_{1,1}$, $\widetilde{\mathbf{p}}_{1,1}$, $\widetilde{\mathbf{q}}_{2,1}$, $\widetilde{\mathbf{p}}_{1,2}$, $\widetilde{\mathbf{q}}_{1,1}$. Then the following points have to be adjusted: $\mathbf{f}_{0,0}$, $\mathbf{f}_{1,0}$, $\mathbf{f}_{2,0}$, $\mathbf{f}_{2,1}$, $\mathbf{f}_{2,2}$, $\mathbf{f}_{1,2}$, $\mathbf{f}_{0,2}$, $\mathbf{f}_{0,1}$, $\mathbf{q}_{1,0}$, $\mathbf{q}_{2,0}$, $\mathbf{p}_{2,1}$, $\mathbf{p}_{2,2}$, $\mathbf{q}_{2,2}$, $\mathbf{q}_{1,2}$, $\mathbf{p}_{0,2}$, $\mathbf{p}_{0,1}$. The points $\mathbf{p}_{0,0}$, $\mathbf{p}_{1,0}$, $\mathbf{p}_{2,0}$, $\mathbf{q}_{3,0}$, $\mathbf{q}_{3,1}$, $\mathbf{q}_{3,2}$, $\mathbf{p}_{2,3}$, $\mathbf{p}_{1,3}$, $\mathbf{p}_{0,3}$, $\mathbf{q}_{0,2}$, $\mathbf{q}_{0,1}$, $\mathbf{q}_{0,0}$ remain unchanged.

All we have to show here is how to adjust the points $\mathbf{f}_{0,0}$, $\mathbf{f}_{2,0}$, $\mathbf{f}_{2,2}$, $\mathbf{f}_{0,2}$ to the points $\widetilde{\mathbf{f}}_{0,0}$, $\widetilde{\mathbf{f}}_{2,0}$, $\widetilde{\mathbf{f}}_{2,2}$, $\widetilde{\mathbf{f}}_{0,2}$. Then the other adjusted points can be geometrically constructed using the properties described in the Figs. 13(b) and 14(b).

The adjusted point $\widetilde{\mathbf{f}}_{0,0}$ lies on $\text{lin}(\mathbf{f}_{0,0}, \mathbf{b}_{1,1})$. Similarly, $\widetilde{\mathbf{f}}_{2,0}$ lies on $\text{lin}(\mathbf{f}_{2,0}, \mathbf{b}_{2,1})$, $\widetilde{\mathbf{f}}_{2,2}$ lies on $\text{lin}(\mathbf{f}_{2,2}, \mathbf{b}_{2,2})$, $\widetilde{\mathbf{f}}_{0,2}$ lies on $\text{lin}(\mathbf{f}_{0,2}, \mathbf{b}_{1,2})$. We present a construction for $\widetilde{\mathbf{f}}_{2,2}$, the points $\widetilde{\mathbf{f}}_{0,0}$, $\widetilde{\mathbf{f}}_{2,0}$, $\widetilde{\mathbf{f}}_{0,2}$ can be constructed in a similar way.

Consider Fig. 21(a). First we construct the auxiliary points

$$\begin{aligned}
 \mathbf{h}_1 &= \frac{w_{3,2} \cdot \mathbf{b}_{3,2} + w_{3,3} \cdot \mathbf{b}_{3,3} + w_{2,3} \cdot \mathbf{b}_{2,3}}{w_{3,2} + w_{3,3} + w_{2,3}}, \\
 \mathbf{h}_2 &= \frac{w_{1,1} \cdot \mathbf{b}_{1,1} + w_{2,1} \cdot \mathbf{b}_{2,1} + w_{1,2} \cdot \mathbf{b}_{1,2}}{w_{1,1} + w_{2,1} + w_{1,2}}, \\
 \mathbf{h}_3 &= \frac{\widetilde{w}_{1,1} \cdot \mathbf{b}_{1,1} + \widetilde{w}_{2,1} \cdot \mathbf{b}_{2,1} + \widetilde{w}_{1,2} \cdot \mathbf{b}_{1,2}}{\widetilde{w}_{1,1} + \widetilde{w}_{2,1} + \widetilde{w}_{1,2}}.
 \end{aligned}$$

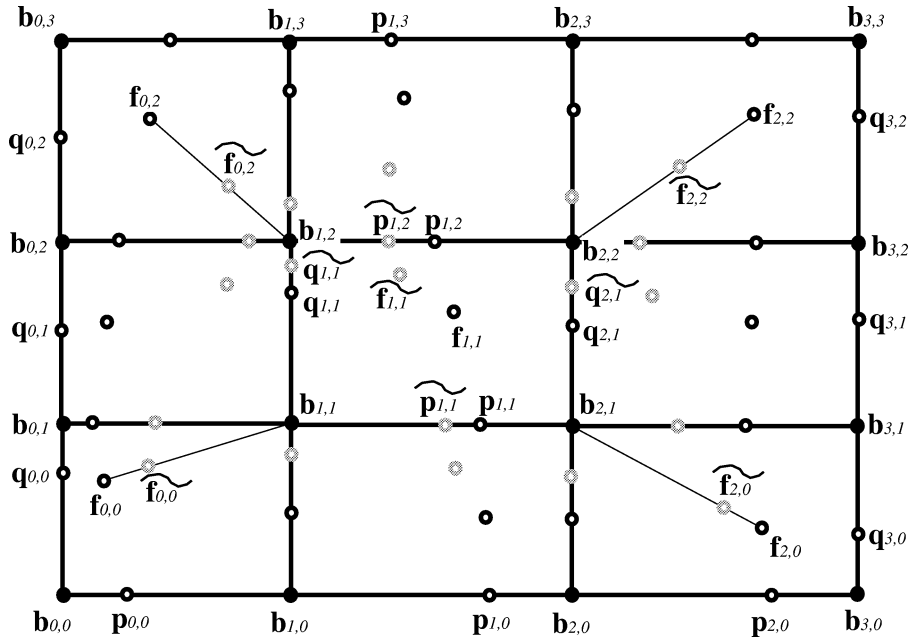


Fig. 20. Moving the Farin points $f_{1,1}$, $p_{1,1}$, $q_{2,1}$, $p_{1,2}$, $q_{1,1}$ to $\widetilde{f}_{1,1}$, $\widetilde{p}_{1,1}$, $\widetilde{q}_{2,1}$, $\widetilde{p}_{1,2}$, $\widetilde{q}_{1,1}$. The adjacent Farin points have to be adjusted.

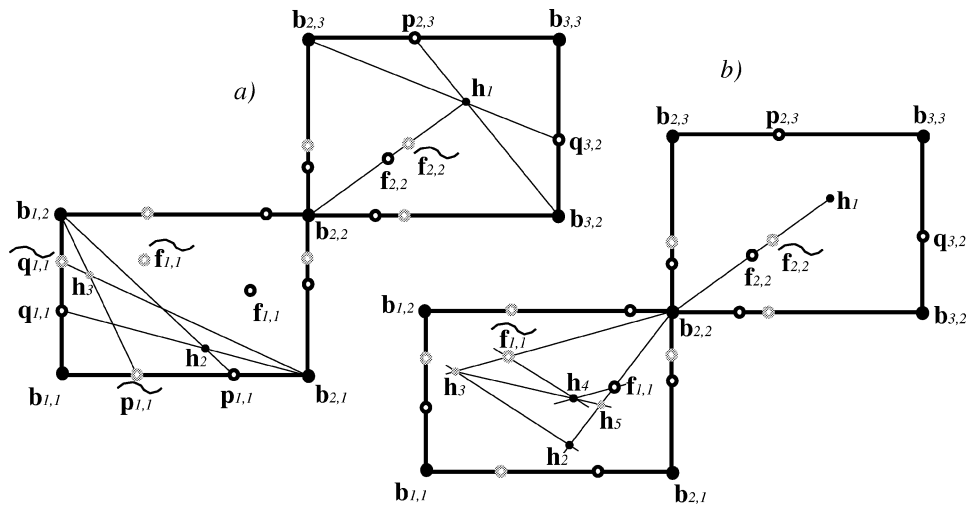


Fig. 21. Moving the Farin points $f_{1,1}$, $p_{1,1}$, $q_{2,1}$, $p_{1,2}$, $q_{1,1}$ to $\widetilde{f}_{1,1}$, $\widetilde{p}_{1,1}$, $\widetilde{q}_{2,1}$, $\widetilde{p}_{1,2}$, $\widetilde{q}_{1,1}$: constructing the adjusted Farin point $\widetilde{f}_{2,2}$. (a) Constructing the auxiliary points h_1, h_2, h_3 ; (b) Constructing the auxiliary points h_4, h_5 . Then $cr(b_{2,2}, f_{1,1}, h_5, h_2) = cr(b_{2,2}, f_{2,2}, \widetilde{f}_{2,2}, h_1)$. This gives the location of $\widetilde{f}_{2,2}$ uniquely.

$\widetilde{w}_{1,1}, \widetilde{w}_{2,1}, \widetilde{w}_{1,2}$ are the new weights after moving $\mathbf{f}_{1,1}, \mathbf{p}_{1,1}, \mathbf{q}_{2,1}, \mathbf{p}_{1,2}, \mathbf{q}_{1,1}$:

$$\mathbf{h}_1 := \text{int}(\text{lin}(\mathbf{b}_{3,2}, \mathbf{p}_{2,3}), \text{lin}(\mathbf{b}_{2,3}, \mathbf{q}_{3,2})),$$

$$\mathbf{h}_2 := \text{int}(\text{lin}(\mathbf{b}_{1,2}, \mathbf{p}_{1,1}), \text{lin}(\mathbf{b}_{2,1}, \mathbf{q}_{1,1})),$$

$$\mathbf{h}_3 := \text{int}(\text{lin}(\mathbf{b}_{1,2}, \widetilde{\mathbf{p}}_{1,1}), \text{lin}(\mathbf{b}_{2,1}, \widetilde{\mathbf{q}}_{1,1})).$$

Then the points $\mathbf{b}_{2,2}, \mathbf{f}_{2,2}, \mathbf{h}_1$ are collinear. $\mathbf{b}_{2,2}, \mathbf{f}_{1,1}, \mathbf{h}_2$ and $\mathbf{b}_{2,2}, \widetilde{\mathbf{f}}_{1,1}, \mathbf{h}_3$ are also collinear. Now consider Fig. 21(b). We construct

$$\mathbf{h}_4 := \text{int}(\text{par}(\text{lin}(\mathbf{b}_{2,2}, \widetilde{\mathbf{f}}_{1,1}), \mathbf{f}_{1,1}), \text{par}(\text{lin}(\mathbf{h}_2, \mathbf{h}_3), \widetilde{\mathbf{f}}_{1,1})),$$

$$\mathbf{h}_5 := \text{int}(\text{lin}(\mathbf{b}_{2,2}, \mathbf{f}_{1,1}), \text{lin}(\mathbf{h}_3, \mathbf{h}_4)).$$

Then it is a straightforward exercise in algebra to show that

$$\text{cr}(\mathbf{b}_{2,2}, \mathbf{f}_{1,1}, \mathbf{h}_5, \mathbf{h}_2) = \text{cr}(\mathbf{b}_{2,2}, \mathbf{f}_{2,2}, \widetilde{\mathbf{f}}_{2,2}, \mathbf{h}_1) = \frac{\widetilde{w}_{2,2}}{w_{2,2}}.$$

From this fact it is a basic construction to get $\widetilde{\mathbf{f}}_{2,2}$ from $\mathbf{b}_{2,2}, \mathbf{f}_{2,2}, \mathbf{h}_1, \mathbf{f}_{1,1}, \mathbf{h}_5, \mathbf{h}_2$ (see, for instance, (Farin, 1995)). The adjusted points $\widetilde{\mathbf{f}}_{0,2}, \widetilde{\mathbf{f}}_{0,0}, \widetilde{\mathbf{f}}_{2,0}$ can be constructed in a similar way. Thus the problem of adjusting the adjacent Farin points geometrically is solved.

3.2. Independent Farin points in a rectangular Bézier point scheme

In this subsection we want to establish a system of independent Farin points for rectangular Bézier point schemes. Here we only consider Bézier patches of the order $n \times n$. The solution consists of a number of extended Farin points and in addition a number of “normal” Farin points $\mathbf{f}_{i,j}$ (i.e., the barycentric combination of the four corners of the subquadrilateral).

Fig. 22(a) shows the solution for a patch of the order $n \times n$ where n is odd. We provide the quadrilateral in the middle of the patch with an extended Farin point. The same is done with the quadrilaterals “on the diagonals” of the patch. For the other patches we provide every second one with a “normal” Farin point. In Fig. 22(a) the patches with a “normal” or extended Farin point are marked with a black closed line. An extended Farin point is marked by the icon similar to Fig. 16, a “normal” Farin point is marked with a hollow dot.

The solution for an even n is more complicated, see Fig. 22(b). Here we provide the patches “on one diagonal” with an extended Farin point. The other extended Farin points are next to the other diagonal. The rest is filled with normal Farin points for every second patch. This way, the weights of two opposite corner Bézier points have to be fixed explicitly. This can be done by using two Farin points on a line, as shown in Fig. 22(b).

Consider Fig. 23. It shows that the weight ratios in a bicubic Bézier patch can be described in terms of five extended Farin points. Since this case often occurs in practical applications, the practical relevance of the approach presented in this subsection is given.

3.2.1. Remarks

- (1) A similar approach as introduced in (Albrecht, 1995) for the triangular case is possible for the rectangular case. Here we have to search for a system of independent

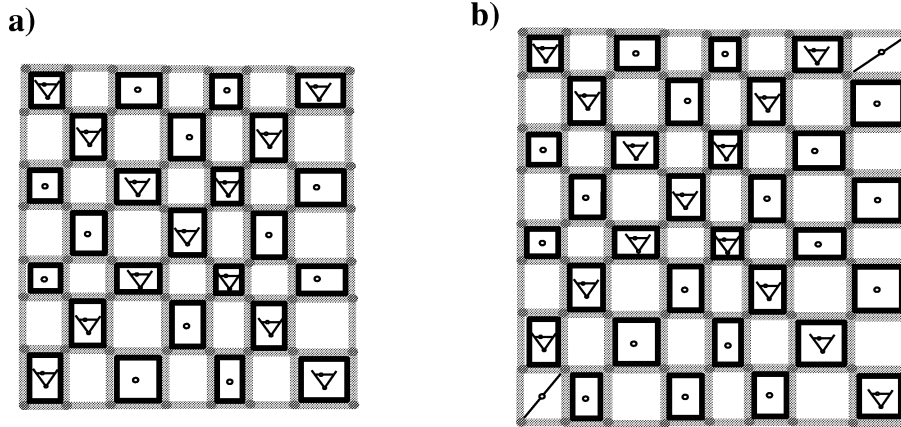


Fig. 22. A system of independent Farin points for a Bézier patch of the order $n \times n$: (a) n odd; (b) n even. The system consists of extended Farin points and normal Farin points. In case (b), the weights of two opposite Bézier points have to be fixed by introducing two Farin points on a line segment.

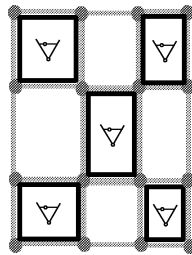


Fig. 23. Describing the weight ratios in a bicubic rational Bézier patch with five extended Farin points.

Farin points $\mathbf{p}_{i,j}$, $\mathbf{q}_{i,j}$ on the line segments of the control net. As in the triangular case, this approach would require more Farin points than the solution introduced in this subsection.

- (2) The solution introduced in this subsection is not the only one. In fact, other systems of extended and normal Farin points are thinkable which solve the problem as well. The distinguishing property of our scheme is the symmetry along at least one “diagonal” in the control net.

4. Conclusions

In this paper we have shown how to use Farin points as a design tool for rational Bézier surfaces on a triangular or rectangular domain. The main problem we had to solve was to keep the contradiction freeness of the systems of Farin points. This was achieved by two

approaches: adjusting the adjacent Farin points or provide only an appropriate subset of Farin points to be movable.

Applying these approaches makes Farin points to a practicable tool for the design of rational Bézier surfaces.

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