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Are isophotes and reflection lines the same?

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Abstract

Isophotes and reflection lines are standard tools for surface interrogation. We study the correlations between them. We show that isophotes and reflection lines are different (but not disjunct) classes of surface curves. Furthermore we introduce the concept of reflection circles as a generalization of isophotes and reflection lines. Reflection circles can be considered as the mirror images of a family of concentric circles on the surface. We show that reflection circles contain both isophotes and reflection lines as special cases. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Surface interrogation; Isophotes; Reflection lines

1. Introduction

Isophotes and reflection lines are well-studied standard tools for surface interrogation. The research on properties of isophotes was done quite independently of the research done on reflection lines. It turned out that both curve classes have similar properties concerning continuity features and fairness for properly shaped surfaces.

Isophotes. An isophote on a surface $\mathbf{x}$ can be defined by an eye point $\mathbf{e}_p$ and an angle α. Then the isophote consists of all surface points $\mathbf{x}$ with the property $\text{angle}(\mathbf{e}_p - \mathbf{x}, \mathbf{n}) = \alpha$ where $\mathbf{n}$ denotes the normalized surface normal. Fig. 1(a) illustrates this.

This definition of isophotes can be simplified by considering an eye direction vector $\mathbf{e}$ instead of an eye point $\mathbf{e}_p$. Then one isophote is defined by $\mathbf{e}$ and a scalar value v. It consists of all surface points with

$$\mathbf{e}\mathbf{n} = v. \tag{1}$$

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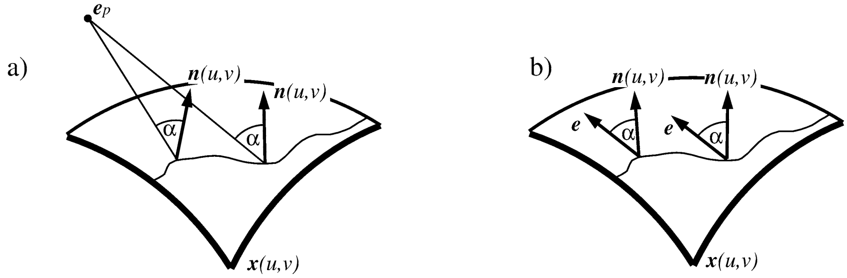

Fig. 1. (a) Definition of isophotes using an eye point $\mathbf{e}_p$; (b) simplified definition of isophotes using an eye direction **e**: all points with a constant angle α between **e** and the surface normal **n** lie on an isophote.

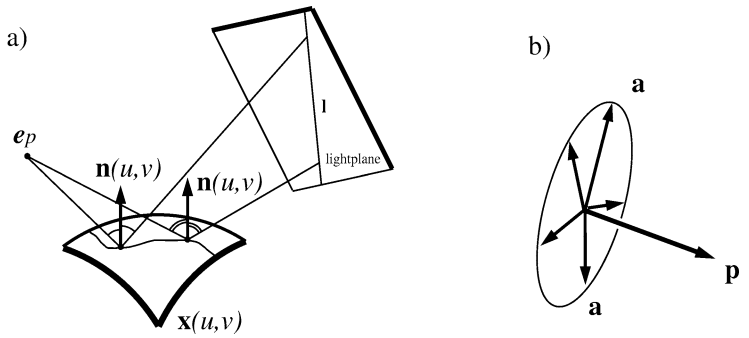

Fig. 2. (a) Definition of reflection lines: a reflection line on the surface **x** is the mirror image of the light line **l** on **x** while looking from $\mathbf{e}_p$. (b) A line at infinity is defined by a vector **p**; it consists of all directions **a** with $\mathbf{ap} = 0$.

This means that all points on the isophote have the same angle between **e** and the surface normal **n** (Fig. 1(b) illustrates).

The definition of isophotes using an eye vector is commonly used in the literature (Hagen et al., 1992; Poeschl, 1984; Theisel, 1997). It has the advantage that the definition of isophotes depends only on the normal and not on the location: two surface points with the same normal direction are located on the same isophote, no matter how **e** is chosen.

A parametric family of isophotes on a surface is obtained by defining the fixed normalized **e** and varying v between -1 and 1. For every parameter v, a particular isophote is defined.

Reflection lines. A reflection line (Hagen et al., 1992; Kaufmann and Klass, 1988; Klass, 1980; Theisel and Farin, 1997) on a surface **x** is defined by an eye point $\mathbf{e}_p$, a certain plane (called light plane), and a line **l** in the light plane (called light line). Considering **x** as a mirror, the reflection line on **x** is defined as the mirror image of **l** on **x** while looking from $\mathbf{e}_p$. Fig. 2(a) gives an illustration. A family of reflection lines is obtained by considering all parallel lines of **l** in the light plane.

It is known that isophotes and reflection lines have a number of properties in common:

- A G^n continuous surface $\mathbf{x}$ ensures both G^{n-1} isophotes and G^{n-1} reflection lines on $\mathbf{x}$.
- Isophotes and reflection lines react rather sensitive on small perturbations of the surface.
- Isophotes and reflection lines cannot be expressed in a closed parametric form but only as a numerical solution of a system of partial differential equations.

However, it seems to be an open question whether or not isophotes and reflection lines are essentially the same curve concept or not. Parts of the literature treat isophotes as a special case of reflection lines (Farin, 1996) while others distinguish carefully between them (Hagen et al., 1992).

In Section 2 of this paper we study the relations between isophotes and reflection lines. It turns out that isophotes and reflection lines are indeed different (but not disjunct) classes of surface curves. Section 3 introduces a new class of surface curves called reflection circles. We show that reflection circles are the generalized concept of both reflection lines and isophotes. Section 4 discusses other concepts of surface curves which are sometimes used instead of isophote and reflection lines. We show that they can be considered as special reflection circles as well.

2. The relation between isophotes and reflection lines

Given the usual definition of isophotes and reflection lines as given in the Figs. 1(b) and 2(a), their comparison is trivial: isophotes and reflection lines are different curve classes. This is due to the fact that reflection lines depend both on the location and the normal direction of a surface point. Given an eye point and a light line, a reflection line on the surface changes while translating the surface. In order to make isophotes and reflection lines comparable, the definition of reflection lines has to be simplified in such a way that they depend only on the surface normals. This simplification is introduced in the next section.

2.1. Reflection lines at infinity

The simplification of isophotes from the definition shown in Fig. 1(a) to the commonly used definition shown in Fig. 1(b) can be interpreted as moving the eye point $\mathbf{e}_p$ to a point $\mathbf{e}$ at infinity. The underlying space of this is an extended 3D affine space which serves as a model of the 3D projective space. This extended affine space additionally consists of points at infinity which are represented by directions. See (Boehm and Prautzsch, 1994) for an introduction of the projective space and the extended affine space.

Similar to isophotes, we simplify the concept of reflection lines by moving both the eye point $\mathbf{e}_p$ and the light line $\mathbf{l}$ to infinity. Doing this, $\mathbf{e}_p$ converges to the eye direction $\mathbf{e}$ while $\mathbf{l}$ converges to a line at infinity. A line at infinity consists of a number of points at infinity, i.e., a set of coplanar directions. This way a line at infinity can be described by a vector $\mathbf{p}$; it consists of all directions $\mathbf{a}$ which are perpendicular to $\mathbf{p}$. Fig. 2(b) gives an illustration.

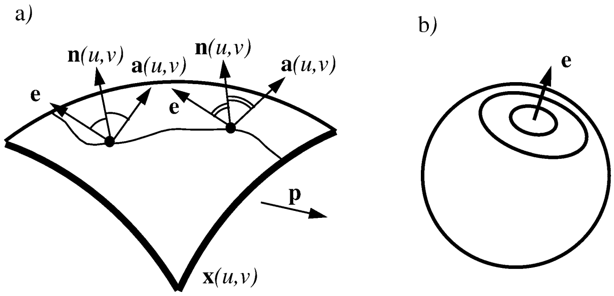

Fig. 3. (a) Definition of reflection lines at infinity by an eye vector **e** and a line at infinity represented by a vector **p**. It consists of all surface points where the reflected eye direction **a** lies on the line **p**, i.e., $\mathbf{ap} = 0$. (b) Isophotes on the unit sphere **s** are circles on **s** perpendicular to **e**.

Using the concept of points and lines at infinity we get the following simplified definition of reflection lines. A reflection line is defined by an eye direction **e** and a line at infinity represented by a vector **p**. It consists of all surface points with the following property: the reflected eye direction **a** lies on the line at infinity, i.e., $\mathbf{ap} = 0$. (The reflected ray is defined by being coplanar to **e** and **n**, and $\text{angle}(\mathbf{e}, \mathbf{n}) = \text{angle}(\mathbf{a}, \mathbf{n})$.) Fig. 3(a) gives an illustration.

A family of reflection lines at infinity is obtained by varying **p** along a line at infinity. Given **e** and **p**, a reflection line at infinity depends only on the surface normals, not on the location on the surface. As in the isophote case, the major properties of reflection lines are preserved by simplifying them to reflection lines at infinity. For the rest of the paper we only consider reflection lines at infinity.

2.2. Isophotes and reflection lines on the unit sphere

For given **e** and **p**, both isophotes and reflection lines depend only on the surface normals and not on the locations of the surface. Two surface points with the same normal direction always lie on the same isophote/reflection line. Thus, in order to compare isophotes and reflection lines, it is sufficient to study their behavior on just one particular surface—the unit sphere.

Given the unit sphere **s**, an isophote (defined by **e** and v) is a circle on **s** perpendicular to **e**. A family of isophotes is a set of concentric circles on **s** which are perpendicular to **e**. Fig. 3(b) illustrates this.

To describe a reflection line on **s**, we formulate the following

Theorem 1. *Given the unit sphere* **s**, *the reflection line defined by the eye direction* **e** (*normalized*) *and the line* **p** *at infinity* (*normalized*) *is the intersection curve of* **s** *with an upright elliptic cylinder* **c**, *where* **c** *is described by the following properties*:

- *the center of* **s** *lies on the center axis of* **c**,
- *the sweeping direction of* **c** *is in the plane defined by* **e** *and* **p**, *and perpendicular to the average of* **e** *and* **p**,
- *the minor axis of the ellipse defining* **c** *is the average of* **e** *and* **p**; *the minor radius* r_1 *is* $r_1 = \frac{\sqrt{2}}{2}$,

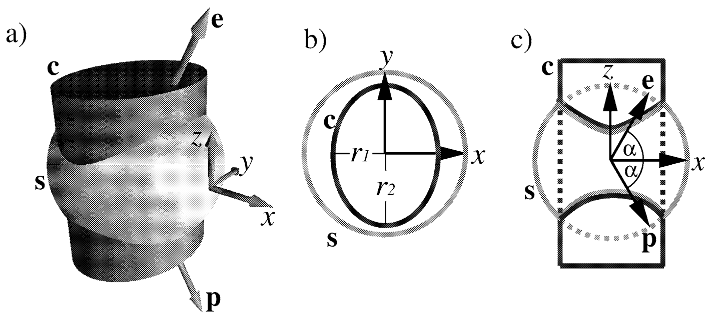

Fig. 4. A reflection line on the unit sphere **s** is the intersection curve of **s** with an elliptic cylinder **c**. (a) 3D image of the configuration; (b) configuration in x–y plane; (c) configuration in x–z plane. Note that the minor radius r_1 of the defining ellipse is always $r_1 = \frac{\sqrt{2}}{2}$.

- *the major axis of the ellipse defining* **c** *is perpendicular to the plane defined by* **e** *and* **p**; *the major radius* r_2 *is* $r_2 = \frac{r_1}{\sin\alpha}$ *where* $\alpha = \frac{\text{angle}(\mathbf{e},\mathbf{p})}{2}$.

Fig. 4 gives an illustration. To prove this theorem, we apply a transformation of the coordinate system in such a way that the origin is the center of **s**, and the coordinate axes **i**, **j**, **k** are defined as

$$\mathbf{i} = \frac{\mathbf{e}+\mathbf{p}}{\|\mathbf{e}+\mathbf{p}\|}, \qquad \mathbf{j} = \frac{\mathbf{e}\times\mathbf{p}}{\|\mathbf{e}\times\mathbf{p}\|}, \qquad \mathbf{k} = \mathbf{i}\times\mathbf{j}. \tag{2}$$

In this new coordinate system, **e** and **p** can be expressed as

$$\mathbf{e} = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} \cos\alpha \\ 0 \\ -\sin\alpha \end{pmatrix}, \tag{3}$$

where $\alpha = \frac{\text{angle}(\mathbf{e},\mathbf{p})}{2}$. The elliptic cylinder **c** can be expressed in the new coordinate system in parametric form as

$$\mathbf{c}(s,t) = \begin{pmatrix} r_1 \cdot \cos t \\ r_2 \cdot \sin t \\ s \end{pmatrix}. \tag{4}$$

Then the intersection curve of **c** and **s** can be expressed in parametric form as

$$\mathbf{x}(t) = \mathbf{n} = \begin{pmatrix} r_1 \cdot \cos t \\ r_2 \cdot \sin t \\ \pm\sqrt{1 - r_1^2 \cdot \cos^2 t - r_2^2 \cdot \sin^2 t} \end{pmatrix}. \tag{5}$$

Setting the reflected ray **a** as

$$\mathbf{a} = 2(\mathbf{en})\mathbf{n} - \mathbf{e} \tag{6}$$

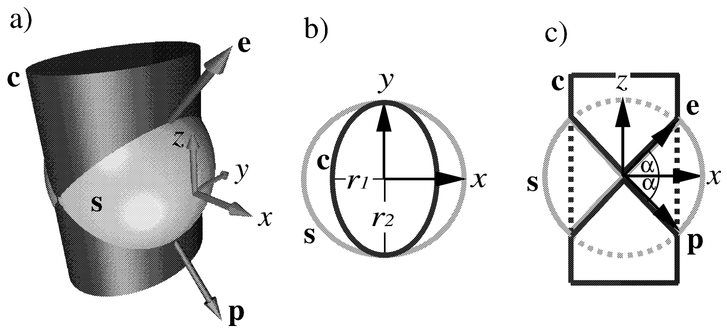

Fig. 5. The reflection line on **s** for $\alpha = 45°$ gives a pair of great circles on **s**. (a) 3D image of the configuration; (b) configuration in x–y plane; (c) configuration in x–z plane.

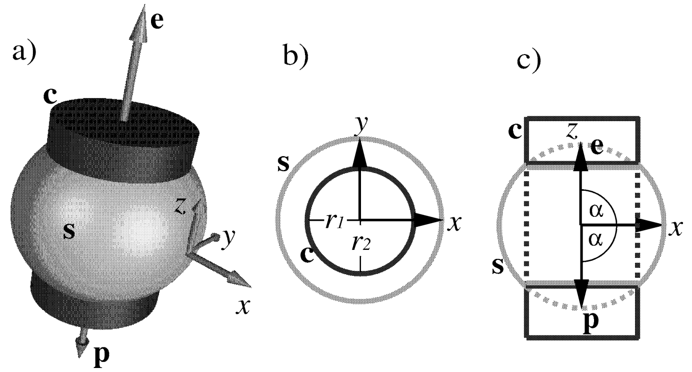

Fig. 6. The reflection line on **s** for $\mathbf{e} = -\mathbf{p}$ is a pair of circles on **s** with the radii $r_1 = r_2 = \frac{\sqrt{2}}{2}$. (a) 3D image of the configuration; (b) configuration in x–y plane; (c) configuration in x–z plane.

and setting

$$r_1 = \frac{\sqrt{2}}{2}, \qquad r_2 = \frac{\sqrt{2}}{2} \cdot \frac{1}{\sin\alpha} \tag{7}$$

it is a straightforward exercise in algebra to obtain $\mathbf{ap} = 0$ from (3), (5), (6), (7). □

Figs. 5–7 show special cases of reflection lines on **s**.

2.3. Comparison between isophotes and reflection lines

From studying isophotes and reflection lines on the unit sphere we can deduce that they are different classes of surface curves. In fact, the only circles which can be obtained as reflection lines have the radius $\frac{\sqrt{2}}{2}$ (for $\alpha = 90°$, see Fig. 6) or 1 (for $\alpha = 45°$, see Fig. 5). All the other circles (i.e., isophotes) on **s** cannot be reflection lines because the minor radius of the ellipse defining **c** is constant $\frac{\sqrt{2}}{2}$. Fig. 8(a) shows the relation between the sets of all isophotes and reflection lines.

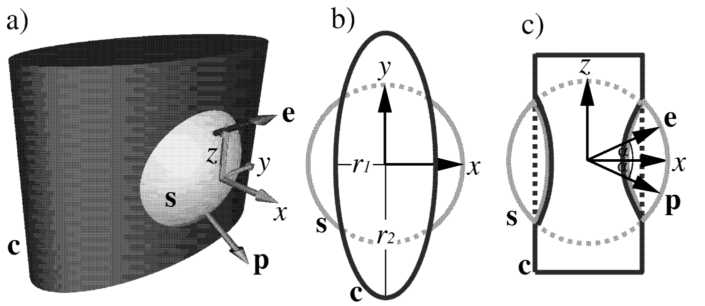

Fig. 7. Reflection line on **s** for $\alpha = 30°$. (a) 3D image of the configuration; (b) configuration in x–y plane; (c) configuration in x–z plane.

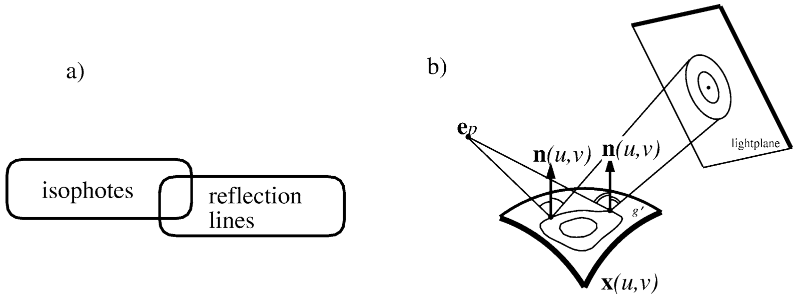

Fig. 8. (a) Set diagram for the sets of all isophotes and reflection lines. (b) Definition of reflection circles: mirror images of concentric circles on the light plane while looking from $\mathbf{e}_p$ to $\mathbf{x}$.

3. Reflection circles

It is the purpose of this section to find a class of surface curves which is a generalization of both reflection lines and isophotes. These curves are called reflection circles.

Reflection circles are defined similarly to reflection lines. Instead of considering parallel lines in the light plane, reflection circles are obtained by considering concentric circles in the light plane. The reflection circles on the surface **x** are the mirror images of the concentric circles in the light plane. Fig. 8(b) illustrates this.

As in the case of isophotes and reflection lines, we simplify the concept of reflection circles by moving both the eye point and the concentric circles to infinity. A circle at infinity is defined by a direction **r** and an angle α. It consists of all vectors **a** (i.e., points at infinity) which fulfill angle$(\mathbf{r}, \mathbf{a}) = \alpha$. Fig. 9(a) illustrates this.

A reflection circle at infinity is defined by the eye direction **e** and the circle at infinity given by **r** and α. It consists of all surface points with the property that the reflected eye direction **a** lies on the circle at infinity, i.e., angle$(\mathbf{r}, \mathbf{a}) = \alpha$. Fig. 9(b) illustrates.

Considering **e** and **n** as being normalized, the reflected eye direction can be computed as (6). Inserting this into $\mathbf{ar} = \cos\alpha$, we obtain

$$\mathbf{ar} = 2(\mathbf{en})(\mathbf{rn}) - (\mathbf{er}) = \cos\alpha. \tag{8}$$

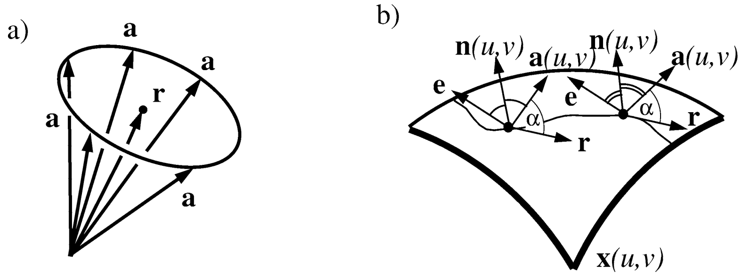

Fig. 9. (a) Circle at infinity, defined by $\mathbf{r}$ and α. It consists of all vectors $\mathbf{a}$ with angle$(\mathbf{r},\mathbf{a}) = \alpha$. (b) Reflection circles at infinity defined by $\mathbf{e}$, $\mathbf{r}$ and α. It consists of all surface points where the reflected eye direction $\mathbf{a}$ lies on the circle at infinity, i.e., angle$(\mathbf{r},\mathbf{a}) = \alpha$.

Setting $v = \frac{\cos\alpha + (\mathbf{er})}{2}$, we obtain: a reflection circle is defined by two vectors $\mathbf{e}, \mathbf{r}$ and a scalar v. It consists of all surface points with the property

$$(\mathbf{en})(\mathbf{rn}) = v. \tag{9}$$

There are several possible ways of defining a family of reflection circles:

(1) vary v,
(2) vary $\mathbf{a}$ along a line at infinity,
(3) apply (1) and (2) simultaneously.

As for isophotes and reflection lines, the definition of reflection circles at infinity depends only on the surface normals and not on the surface locations. Thus it is sufficient to study reflection circles on the unit sphere.

3.1. Reflection circles on the unit sphere

To describe reflection circles on the unit sphere, we give the following

Theorem 2. *Given is the reflection circle* $\mathbf{x}$ *on the unit sphere* $\mathbf{s}$ *defined by the normalized vectors* $\mathbf{e}, \mathbf{r}$ *and the scalar* v. *Then* $\mathbf{x}$ *is the intersection curve of* $\mathbf{s}$ *with an upright elliptic cylinder* $\mathbf{c}$ *with the following properties*:

- *the center of* $\mathbf{s}$ *lies on the center axis of* $\mathbf{c}$,
- *the sweeping direction of* $\mathbf{c}$ *is the average of* $\mathbf{r}$ *and* $\mathbf{e}$,
- *the minor direction of the ellipse defining* $\mathbf{c}$ *lies in the plane defined by* $\mathbf{r}$ *and* $\mathbf{e}$; *the minor radius* r_1 *is* $r_1 = \sqrt{\frac{1+(\mathbf{er})}{2} - v}$,
- *the major direction of the defining ellipse is perpendicular to* $\mathbf{r}$ *and* $\mathbf{e}$; *the major radius* r_2 *is* $r_2 = \frac{r_1}{\sin\alpha}$ *with* $\alpha = 90^\circ - \frac{\text{angle}(\mathbf{r},\mathbf{e})}{2}$.

Fig. 10 illustrates the theorem. To prove Theorem 2, we apply a transformation of the coordinate system in such a way that the origin is the center of $\mathbf{s}$, and the coordinate axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are defined as

$$\mathbf{k} = \frac{\mathbf{e}+\mathbf{r}}{\|\mathbf{e}+\mathbf{r}\|}, \qquad \mathbf{j} = \frac{\mathbf{r}\times\mathbf{e}}{\|\mathbf{r}\times\mathbf{e}\|}, \qquad \mathbf{i} = \mathbf{j}\times\mathbf{k}. \tag{10}$$

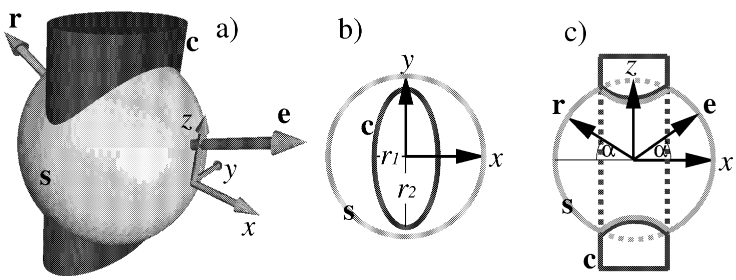

Fig. 10. A reflection circle on the unit sphere **s** is the intersection curve with an elliptic cylinder **c**. (a) 3D view, (b) projection into the x–y plane, (c) projection into the x–z plane. Both the minor radius r_1 and the major radius r_2 can be varied by varying $\mathbf{e}$, $\mathbf{r}$ and v.

In this new coordinate system, **e** and **r** can be expressed as

$$\mathbf{e} = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} -\cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix}, \tag{11}$$

where $\alpha = 90° - \frac{\text{angle}(\mathbf{r},\mathbf{e})}{2}$. The elliptic cylinder **c** can be expressed in the new coordinate system in parametric form by (4). Then the intersection curve of **c** and **s** can be expressed in parametric form as (5). Setting

$$r_1 = \sqrt{\frac{1+(\mathbf{er})}{2} - v}, \qquad r_2 = r_1 \cdot \frac{1}{\sin\alpha} \tag{12}$$

we get from (11), (5) and (12):

$$(\mathbf{en})(\mathbf{rn}) = v \tag{13}$$

which proves that $\mathbf{x}(t)$ defined by (5) and (12) is a reflection circle. □

Note that the minor radius r_1 of the ellipse defining **c** can be chosen freely between 0 and 1, while the major radius r_2 can be chosen freely between r_1 and infinity. For all choices of r_1 and r_2, appropriate $\mathbf{e}$, $\mathbf{r}$, v can be found to describe the intersection curve of **c** and **s** as a reflection circle.

3.2. Relation between reflection circles, isophotes and reflection lines

The relation between reflection circles, isophotes and reflection lines can be formulated in

Theorem 3. *Given is a reflection circle* $\mathbf{x}$ *on a surface which is described by the normalized eye direction* $\mathbf{e}$*, the (normalized) center* $\mathbf{r}$ *of the circle at infinity, and the scalar value* v*. Then the following statements apply*:

(1) $\mathbf{x}$ *is an isophote iff* $\mathbf{r} = \mathbf{e}$ *or* $\mathbf{r} = -\mathbf{e}$.
(2) $\mathbf{x}$ *is a reflection line iff* $\mathbf{re} = 2v$.

Proof. The proof of (1) comes from (1) and (9). (2) Follows directly from (7) and (12). □

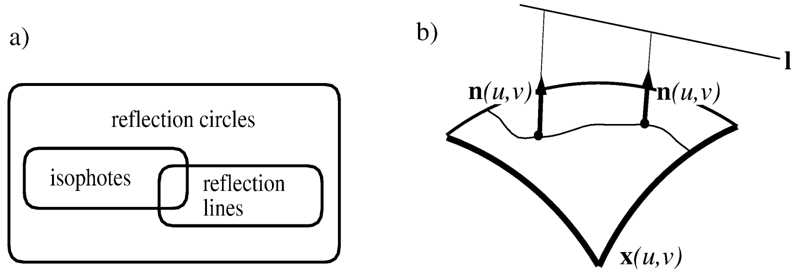

Fig. 11. (a) Set diagram for reflection circles, isophotes and reflection lines; (b) definition of a light line on the surface **x** by a straight line **l**.

The correlations between reflection circles, isophotes and reflection lines are illustrated in Fig. 11(a).

A family of isophotes is obtained by setting $\mathbf{r} = \pm\mathbf{e}$ and varying v. A family of reflection lines is obtained by varying **r** and v simultaneously. In fact, **r** is moved along a line at infinity while v is adjusted in such a way that $2v = \mathbf{re}$.

4. Further concepts of surface curves

In this section we study further concepts of surface curves concerning their relation to reflection circles.

Light lines. A light line on a surface is defined by a straight line **l**. It consists of all surface points **x** with the property that the ray $\mathbf{x} + \lambda\mathbf{n}$ intersects **l**. Fig. 11(b) illustrates this.

Simplifying the concept of light lines by moving **l** to infinity, we obtain: a light line at infinity is defined by a line **r** at infinity. It consists of all surface points with $\mathbf{nr} = 0$.

A light line on the unit sphere **s** is the great circle perpendicular to **r**. Thus light lines are special cases of reflection circles.

Isophenges. Isophenges (Hoschek and Lasser, 1989) are defined as lines of constant apparent light intensity on a surface. Given a surface, a light direction and a view direction, an isophenge consists of all surface points with $\cos\lambda\cos\theta = \text{const}$. Here λ is the angle between light direction and surface normal while θ is the angle between view direction and surface normal. Thus isophenges and reflection circles are essentially the same.

Simplified reflection lines following (Kaufmann and Klass, 1988). In (Kaufmann and Klass, 1988) another class of simplified reflection lines is used. Here families of lines on the surface are considered instead of the surface itself. The angle between the tangents of these line families and a certain vector **e** is used to determine reflection lines. The families of surface curves can be parametric lines or intersections with planes. For our purposes we are interested only in families of surface curves which are independent of a particular surface parametrization; we consider intersection curves with a family of parallel planes.

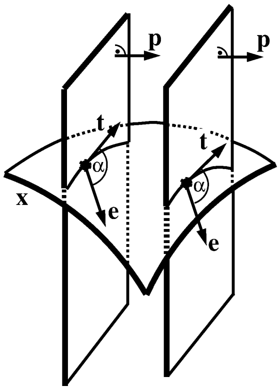

Fig. 12. Simplified reflection lines as used in (Kaufmann and Klass, 1988). Intersection curves with a family of planes parallel to **p** are considered. A reflection line consists of all surface points with a constant angle α between the tangent vector **t** of the intersection curves and the eye direction **e**.

Given is a surface **x**, an eye vector **e**, and a family of parallel planes defined by the common normal **p**. Then all points of **x** with a constant angle α between **e** and the tangent of the intersection curve through this point belong to one reflection line. Fig. 12 illustrates this.

To study this version of reflection lines on the unit sphere **s**, we transform the coordinate system in such a way that the origin is in the center of **s** and $\mathbf{e} = (0, \cos\alpha, \sin\alpha)^{\mathrm{T}}$ and $\mathbf{p} = (0, 1, 0)^{\mathrm{T}}$. Then we get for a point $(x, y, z)^{\mathrm{T}} \in \mathbf{s}$ (i.e., $x^2 + y^2 + z^2 = 1$) the following normalized tangent vector **t**:

$$\mathbf{t} = \begin{pmatrix} \frac{-z}{\sqrt{x^2+z^2}} \\ 0 \\ \frac{x}{\sqrt{x^2+z^2}} \end{pmatrix} = \begin{pmatrix} \frac{-z}{\sqrt{1-y^2}} \\ 0 \\ \frac{x}{\sqrt{1-y^2}} \end{pmatrix}. \tag{14}$$

The points which fulfill $\mathbf{te} = v =$const lie on the intersection of **s** and an elliptic cylinder **c** described by (4) and $r_1 = \frac{v}{\sin\alpha}$, $r_2 = 1$. This describes a pair of great circles on **s**. Thus reflection lines in (Kaufmann and Klass, 1988) are special cases of reflection circles as well.

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