# Combining Topological Simplification and Topology Preserving Compression for 2D Vector Fields

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## Abstract

*Topological simplification techniques and topology preserving compression approaches for 2D vector fields have been developed quite independently of each other. In this paper we propose a combination of both approaches: a vector field should be compressed in such a way that its important topological features (both critical points and separatrices) are preserved while its unimportant features are allowed to collapse and disappear. To do so, a number of new solutions and modifications of pre-existing algorithms are presented. We apply the approach to a flow data set which, is both large and topologically complex, and achieve significant compression ratios there.*

## 1. Introduction

Topological methods have had become a standard tool for visualizing 2D vector fields because they give the opportunity to represent even complex flow structures by only a small number of graphical primitives. The topological skeleton of a vector field essentially consists of critical points (i.e. isolated points where the flow vanishes) and special stream lines called separatrices which separate the flow into areas of different flow behavior. After the introduction of topological methods as a visualization tool in [4], a number of extensions and modifications of topological concepts have been introduced. ([7] , [13] , [1], [15], [5], [14]).

The flow data sets to be visually analyzed are constantly growing both in size and complexity. To deal with this problem, two general approaches have been developed which make use of topological concepts: topological simplification and topology preserving compression of vector fields.

*Topology simplification methods* are motivated in the assumption that not all topological features of a vector field have the same importance. This happens if some of the critical points and separatrices result from noise in the vector field. The simplest way to solve this problem is to apply a smoothing of the vector field before extracting the topology ([2]). More involved techniques repeatedly apply local modifications of the skeleton and/or the underlying vector field in order to remove unimportant critical points. ([1], [2], [12], [11]). They are based on the index theorem for vector fields ([3]) which ensures that the sum of the indices of the critical points remains constant in the modified area.

*Topology preserving compression* techniques can be considered as a somehow opposite approach to topology preserving simplification techniques. Here, the complete topological skeleton is considered to be very important, and compression techniques for the vector field are searched which preserve this topological skeleton completely. [6] is a first approach to compress a vector field under the consideration of preserving certain characteristics of critical points. In [8] and [9], two approaches are presented which do not only preserve the critical points but also the behavior of the separatrices.

Although topology simplification methods and topology preserving compression techniques are somehow opposite approaches which had been developed independently of each other, for real life data sets both problems appear simultaneously. In fact, a data set may have unimportant topological features due to noise, but has to be compressed under preservation of the important topological structures. This gives the main motivation for this paper: to explore how to combine topological simplification techniques and topology preserving compression techniques. Therefore, the goal of the paper is to find a compression technique which preserves important topological structures but removes the unimportant ones. It turns out that for doing so, none of the pre-existing techniques for simplification and compression can be applied as they are, but modifications and even completely new approaches are necessary.

The rest of the paper is organized as follows: section 2 describes the main idea of our approach. Sections 3 and 4 describe the two subproblems to be solved for our approach. Section 5 presents the results.

## 2 Description of the main idea

In this paper we consider piecewise linear vector fields with separatrices starting from saddle points and boundary switch points ([1]). Our approach of combining topological simplification and topology preserving compression can be written as follows:

1. Extract the complete topological skeleton (i.e., all critical points, boundary switch points and separatrices) of the original vector field.
2. Assign a weight $w \in [0,1]$ to every critical point and every separatrix. This weight describes the importance of the critical point or the separatrix.
3. Pick a threshold $w_0 \in [0,1]$ which distinguishes between important ($w \geq w_0$) and unimportant features.
4. Apply a compression of the vector field which ensures the preservation of the important topological features.

As a result of this algorithm, we obtain a compressed version of the original vector field in which the important topological features are preserved. Since for step 1 standard methods for detecting critical points and integrating separatrices are applied, and step 3 is subject of interaction to steer the process, the new parts of the algorithm are in the steps 2 and 4. We discuss step 2 in detail in section 3 while step 4 is treated in section 4.

## 3 Creating a system of importance weights

The problem of creating a system of importance weights for the topological features is strongly related to the problem of topological simplification of vector fields: in both approaches important features have to be depicted. However, we need to compute the importance of both critical points and separatrices. Since all pre-existing topology simplification algorithms only consider critical points, we have to introduce a new algorithm.

The main problem to provide a system of weights is to make it topologically consistent for every threshold $w_0$. This means, for every $w_0 \in [0,1]$ the subskeleton consisting of all critical points and separatrices with a weight larger or equal $w_0$ must describe a valid topological structure. Thus, the following conditions have to be fulfilled:

1. Fulfill the index theorem for the whole vector field: the sum of the indices of all important critical points must be constant and independent of $w_0$.
2. Consistency of separatrices: every separatrix must start in a saddle or boundary switch point, every separatrix must end in a source/sink or leave the domain of the vector field, from every saddle exactly four separatrices and emanating.

To fulfill condition 1, we first group the critical points to pairs such that each pair consists of a saddle and a non-saddle (i.e., source or sink). Then both points of a pair are assigned with the same weight. Section 3.1 discusses this step in detail. To fulfill condition 2, a system of weights for the separatrices has to be found as well as the initial weights of the pairs of critical points have to be updated. Section 3.2 presents the details of this step. Some results of the approach are shown in section 3.3.

### 3.1 Coupling critical points and finding initial weights

Several approaches to couple critical points are reported in the literature. [2] uses the Euclidian distance of the critical points to couple them, [12] demands that critical points building a pair have a common separatrix. However, these coupling strategies do not always yield unique solutions especially in areas containing many critical points. Moreover, they do not provide an importance weight of a critical point. Because of this, we propose an alternative approach which is based on the concept of feature flow fields [10] which originally had been developed to track critical points in time-dependent vector fields. Given a vector field $\mathbf{v}(x,y) = (u(x,y), v(x,y))^T$, we construct a vector field $\mathbf{v}_s$ by applying a very strong Laplacian smoothing which only keeps the boundary of the domain of $\mathbf{v}$ unchanged. This operator can be expected to smooth out many topological features of $\mathbf{v}$. In fact, we expect $\mathbf{v}_s$ to have significantly less critical points than $\mathbf{v}$. Now we consider the time-dependent vector field $\mathbf{u}(x,y,t) = (1-t)\mathbf{v} + t\mathbf{v}_s$ in the time interval $0 \leq t \leq 1$. To track the behavior of the critical points of $\mathbf{u}$ over the time, we construct a 3D vector field $\mathbf{f}$ in such a way that the paths of the critical points of $\mathbf{u}$ over time correspond to stream lines of $\mathbf{f}$. In [10] it is shown that

$$\mathbf{f}(x,y,t) = \begin{pmatrix} \det(\mathbf{u}_y, \mathbf{u}_t) \\ \det(\mathbf{u}_t, \mathbf{u}_x) \\ \det(\mathbf{u}_x, \mathbf{u}_y) \end{pmatrix}. \quad (1)$$

To check if a critical point $(x_0, y_0)$ in $\mathbf{v}$ has a partner critical point, we simply integrate the stream line of $\mathbf{f}$ starting from $(x_0, y_0, 0)$ until $\mathbf{f}$ leaves the valid domain either in a point $(x_1, y_1, 0)$ or in a point $(x_1, y_1, 1)$. In the first case it can be shown ([10]) that $(x_1, y_1)$ is a critical point of $\mathbf{v}$ with an index opposite to $(x_0, y_0)$: $(x_0, y_0)$ and $(x_1, y_1)$ become a couple, and their common weight $w$ is determined by the maximal $t$-value of the stream line of $\mathbf{f}$ between $(x_0, y_0, 0)$ and $(x_1, y_1, 0)$. If the stream line of $\mathbf{f}$ ends in a point $(x_1, y_1, 1)$, $(x_0, y_0)$ does not have a partner critical point in $\mathbf{v}$, it gets the weight 1. Figure 1a gives an illustration. Figure 1b illustrates the approach for the test data set described in section 5.

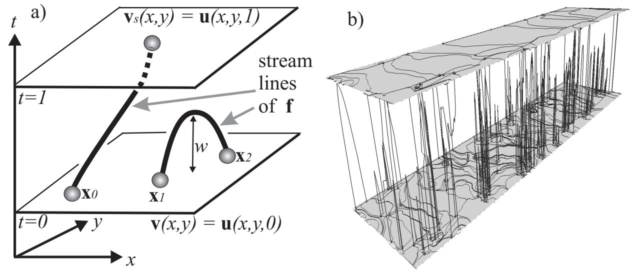


**Figure 1.** a) 3 critical points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ of $\mathbf{v}$ and the stream lines of the feature flow field $\mathbf{f}$; $\mathbf{x}_0$ gets the weight 1 while $\mathbf{x}_1$ and $\mathbf{x}_2$ get the common weight $w$; b) feature flow field for test data set.

### 3.2 Making the weights consistent

Given the partner relation and the initial weights of the critical points from the algorithm described above, we now have to adjust these weights and find weights for the separatrices in such a way that this system is consistent for every threshold $w_0$. This means, the following conditions have to be fulfilled:

1. Two partner critical points must have the same weight. This ensures the index theorem for every $w_0$.
2. The four separatrices starting from a saddle point must have the same weight as their creating saddle.
3. If a separatrix ends in a source or a sink, the weight of this source/sink must not be smaller than the weight of the separatrix.

We start out with the following initial weights: every critical point is assigned the weight from section 3.1, every separatrix starting from a boundary switch point gets a weight of 1, and every separatrix starting from a saddle gets a weight of 0. Then we repeatedly correct the weights which contradict to the conditions above:

a) If two partner critical points have a different weight, they are set to the maximum of both weights.

b) If a separatrix starting from a saddle has a smaller weight than the saddle, its weight is set to the weight of the saddle.

c) If a separatrix ends in a source/sink and the weight of the source/sink is smaller than the weight of the separatrix, the weight of the source/sink is set to the weight of the separatrix.

This process stops if all conditions of 1–3 mentioned above are fulfilled. Obviously, the termination of this algorithm is ensured: in the worst case the algorithm stops when all weights reach the value 1.

### 3.3 Results of creating importance weights

We show the results of creating the importance weights for the skin friction data set which is described later on in section 5. This data set consists of 338 critical points, 34 boundary switch points and 714 separatrices. After finding pairs of critical points and attaching them with weights as described in section 3.1, we analyzed the distribution of the weights on all critical points. The solid curve in figure 2 shows the result of this analysis: Approximately 22% of

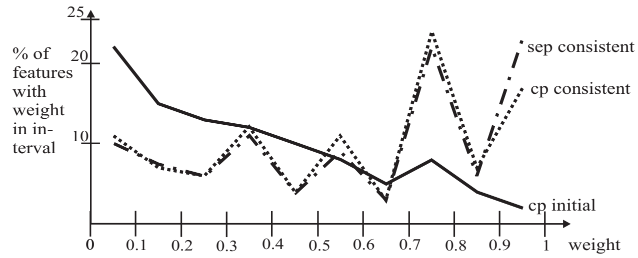


**Figure 2.** Distribution of the importance weights for the skin friction data set; shown are the percentages of the features with a weight in the interval $(i/10, (i+1)/10)$ for $i = 0, .., 9$; solid line: critical points after initializing; dotted line: critical points after making the weights consistent; dash-dotted line: separatrices after making the weights consistent.

all critical points have a very low weight (between 0 and 0.1) while only approximately 2% of the critical points have a very high weight (between 0.9 and 1). The distribution between these extreme values is approximately linear.

In the next step we initialized the separatrices and made the weights consistent as explained in section 3.2. Then the distribution of the weights of the critical points (dotted line) and separatrices (dash-dotted line) is shown in figure 2.

Figure 3 shows the important topological features of the data set for different thresholds $w_0$. The upper image ($w_0 = 0$) shows the complete topological skeleton of the data set while the lowest image ($w_0 = 1$) shows only the topological features with a weight of 1.

## 4 The compression algorithm

To compress the vector field, we use an adapted version of the algorithm in [9] because it seems to be the first and currently only approach which gives reasonable compression ratios while preserving both the critical points and the separatrices. However, for our purposes this algorithm needs some modifications, since now we distinguish between important and unimportant critical points and separatrices.

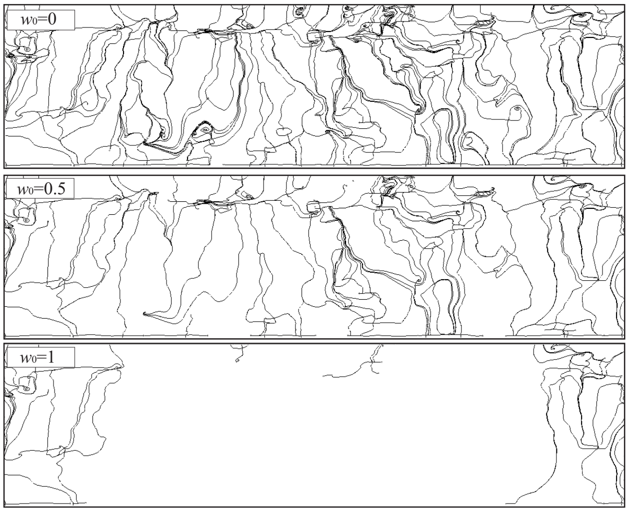


**Figure 3.** Important topological features for different thresholds $w_0$; the upper image ($w_0 = 0$) shows the complete topological skeleton.

In [9] it was shown that – loosely spoken – for a local modification of the vector field it can be decided entirely by a local analysis whether or not this modification changes the topology of the vector field. This property was used to repeatedly apply topology preserving local modifications to a piecewise linear vector field – namely half-edge collapses. To check if a half-edge collapse $\mathbf{p}_0 \rightarrow \mathbf{p}_1$ (where $\mathbf{p}_0$ and $\mathbf{p}_1$ are two vertices sharing a common edge) preserves the topology, a number of points is collected on the boundary of the 1-ring around $\mathbf{p}_0$. These points are the entry and exit points of all separatrices passing through the 1-ring, and the boundary switch points of the vector field restricted to the 1-ring. After simulating the half-edge collapse and recomputing these points on the boundary, the cyclic order of the corresponding points before and after the collapse is compared: if this cyclic order remains unchanged, the half-edge collapse does not change the topology of the vector field. If a half-edge collapse is found to be allowed and if it is carried out, all separatrices passing through the affected area have to be updated.

This algorithm in [9] was designed in such a way that it did not allow any changes of critical points: if a critical point is located inside the 1-ring around $\mathbf{p}_0$, a half-edge collapse $\mathbf{p}_0 \rightarrow \mathbf{p}_1$ was not allowed. For our purposes, this condition has to be relaxed such that only the presence of an important critical point (i.e. a critical point with a weight above the threshold) prohibits the collapse. If unimportant critical points are inside the 1-ring, they are allowed to move, change and even collapse with other unimportant critical points[1].

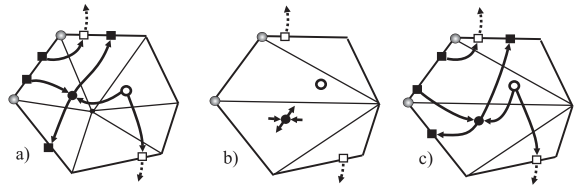


**Figure 4.** a) 1-ring containing an unimportant saddle (solid circle) and an unimportant source (hollow circle); b) simulated half-edge collapse with new locations of the inner critical points; c) integrating the separatrices gives a similar cyclic order of the corresponding points like in a): the half-edge collapse does not change the topology.

If the 1-ring contains unimportant critical points and they do not disappear during the simulated half-edge collapse, the test is very similar to [9]. If a saddle point is present, its new location after the half-edge collapse has to be extracted, and the new separatrices starting from it have to be integrated until they end in a source/sink or leave the 1-ring. Figure 4 gives an illustration.

If the half-edge collapse leads to a collapsing and disappearing of a saddle and a non-saddle (both unimportant critical points), the test of [9] has to be modified such that the separatrices coming from he saddle are not considered for the test. Figure 5 gives an illustration.

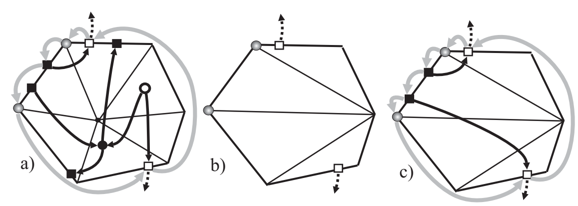


**Figure 5.** a) 1-ring containing two unimportant critical points; b) if a half-edge collapse makes the critical points disappear, the separatrices starting from the saddle point are not considered for comparing the cyclic order (grey arrows) in a); c) reintegrating the remaining separatrices and comparing the cyclic order with a): here the half-edge collapse changes the topology because the cyclic orders differ.

Note that, although we could distinguish between important and unimportant critical points in the algorithm, this distinction cannot be done for separatrices. In fact, all sep-

[1] In fact, a collapse of unimportant critical points is even desired, since it increases the chances to find more allowed half-edge collapses.

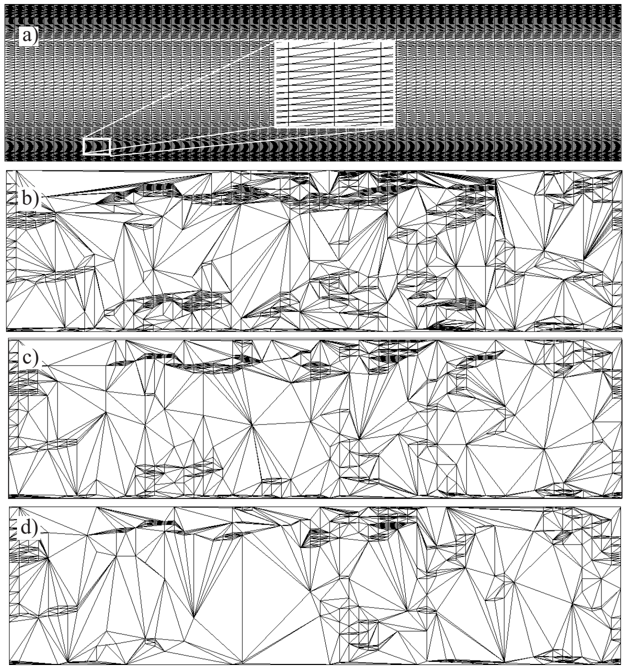


**Figure 6.** Triangular grids of our algorithm: a) original data set (12,726 triangles); b) result from [9] (2,153 triangles); c) new algorithm with $w_0 = 0.5$ (1,367 triangles); d) new algorithm with $w_0 = 1$ (1,040 triangles).

aratrices passing through a 1-ring have to be equally treated as important. This is due to the fact that the theoretical foundation of the algorithm (see [9]) is based on the assumption that the complete topological information about the vector field is present for the local analysis. However, by applying half-edge collapses which remove unimportant critical points, unimportant separatrices are removed as well.

## 5 Results

We applied our approach to a data set which describes the skin friction on a face of a cylinder. It was obtained by a numerical simulation of a flow around a square cylinder[2]. The same data set has been analyzed both for topology simplification ([1]) and topology preserving compression ([6], [9]). This piecewise linear vector field consisting of 12,726 triangles has 338 critical points, 34 boundary switch points, and 714 separatrices. Therefore, it can be considered as a vector field of both a large size and a complex topology.

Figure 6a shows the underlying triangular grid at different steps of our algorithm. Figure 6a shows the triangular grid of the original data set. Figure 6b shows the result of the topology preserving compression technique from [9] consisting of 2,153 triangles. Figure 6c shows the result of our algorithm for $w = 0.5$. Here we obtained 1,367 triangles and a compression ratio of 89.2% at a computing time of 424 seconds on an Intel Xeon 1.7 GHz processor. For $w = 1$, we achieved 1,040 triangles (compression ratio of 91.8%) at a computing time of 472 seconds. This is shown in figure 6d.

Figures 6 shows that for the test data set our approach can significantly decrease the number of obtained triangles (and thus increase the compression ratio) in comparison to [9] if important and unimportant topological features are treated in a different way as explained in section 4. Our algorithm has full control over important features, since they are guaranteed to be preserved. However, the algorithm does not have control over the unimportant features, since they may move and disappear during the compression. In fact, the example has shown that parts of the unimportant features disappear while others only change their location. It would be useful to have a compression algorithm which removes all unimportant topological features, but we have to consider this as an unsolved problem.

[2]The data set was generated by R.W.C.P. Verstappen and A.E.P. Veldman of the University of Groningen (the Netherlands). The authors thank Wim de Leeuw for providing this test data set.

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