
ON THE OBJECTIVITY AND QUASI-OBJECTIVITY OF TSE AND TRA

A PREPRINT

✉ **Holger Theisel**

Department of Computer Science
University of Magdeburg
Magdeburg, Germany
theisel@ovgu.de

✉ **Anke Friederici**

Department of Computer Science
University of Magdeburg
Magdeburg, Germany
anke@isg.cs.uni-magdeburg.de

✉ **Tobias Günther**

Department of Computer Science
Friedrich-Alexander-University Erlangen-Nürnberg
Erlangen, Germany
tobias.guenther@fau.de

ABSTRACT

We analyze two recently-introduced flow measured that are based on a single trajectory only: trajectory stretching exponent (TSE) to detect hyperbolic (stretching) behavior, and trajectory angular velocity (TRA) to detect elliptic (rotation) behavior. Haller et al. [2021] and Haller et al. [2022] introduced TSE, TRA as well as the concept of quasi-objectivity, and formulated theorems about the objectivity and quasi-objectivity of TSE and TRA.

In this paper, we present two counter-examples showing that all theorems in Haller et al. [2021] and Haller et al. [2022] are incorrect.

Keywords Objectivity · Flow Analysis

1 Introduction

Recently, Haller et al. [2021] and Haller et al. [2022] introduced measures based on a single trajectory only. For this, the concept of quasi-objectivity is introduced: Contrary to classical objectivity where a scalar value must be invariant under arbitrary time-dependent Euclidian transformations, for quasi-objectivity a condition (A) is introduced, and invariance is not demanded for all Euclidean transformations but only for those fulfilling (A). Then, Haller et al. [2021] introduced several measures based on a single trajectory: *extended trajectory stretching exponents* TSE and $\overline{\text{TSE}}$, and *extended trajectory angular velocity* TRA, $\overline{\text{TRA}}$. Haller et al. [2021] claimed that TSE and $\overline{\text{TSE}}$ are objective in the extended phase space, and that $\overline{\text{TRA}}$ is quasi-objective in the extended phase space under a certain condition put to the average vorticity in a certain neighborhood of the trajectory. The new measures have been applied in several follow-up papers: Encinas-Bartos et al. [2022], Aksamit and Haller [2022].

In this paper, we show that the claims by Haller et al. [2021] concerning objectivity of TSE, $\overline{\text{TSE}}$, $\overline{\text{TRA}}$ are incorrect. In fact, we show by a counter-example that neither TSE nor $\overline{\text{TSE}}$ are objective in the extended phase space. Further, $\overline{\text{TRA}}$ is not quasi-objective in the extended phase space under an averaged-vorticity-based condition.

In an erratum, Haller et al. [2022] reformulate claims about quasi-objectivity of TSE, TRA. We show by a second counter-example that the new claims in the erratum Haller et al. [2022] are incorrect as well.

In summary, our two counter-examples show that all theorems in Haller et al. [2021] and Haller et al. [2022] are incorrect.

We note that the first counter-example was already published in Theisel et al. [2022], and Haller et al. [2022] was published as a reaction on Theisel et al. [2022].

2 TSE, TRA, and objectivity

Objectivity, a concept from continuum mechanics, refers to the invariance of a measure under a moving reference system. Let $s(\mathbf{x}, t)$, $\mathbf{w}(\mathbf{x}, t)$, $\mathbf{T}(\mathbf{x}, t)$ be a time-dependent scalar field, vector field and tensor field, respectively. Further, let $\tilde{s}(\tilde{\mathbf{x}}, t)$, $\tilde{\mathbf{w}}(\tilde{\mathbf{x}}, t)$, $\tilde{\mathbf{T}}(\tilde{\mathbf{x}}, t)$ be their observations under the Euclidean frame change

$$\mathbf{x} = \mathbf{Q}(t) \tilde{\mathbf{x}} + \mathbf{b}(t) \quad (1)$$

where $\mathbf{Q} = \mathbf{Q}(t)$ is a time-dependent rotation tensor and $\mathbf{b}(t)$ is a time-dependent translation vector. Then s , \mathbf{w} , \mathbf{T} are *objective* if the following conditions hold, cf. Truesdell and Noll [1965]:

$$\tilde{s}(\tilde{\mathbf{x}}, t) = s(\mathbf{x}, t), \quad \tilde{\mathbf{w}}(\tilde{\mathbf{x}}, t) = \mathbf{Q}^T \mathbf{w}(\mathbf{x}, t), \quad \tilde{\mathbf{T}}(\tilde{\mathbf{x}}, t) = \mathbf{Q}^T \mathbf{T}(\mathbf{x}, t) \mathbf{Q}. \quad (2)$$

2.1 TSE and TRA in a nutshell

Haller et al. [2021] introduced measures for stretching and rotation that are based on single trajectories only: *Extended trajectory stretching exponents* TSE, $\overline{\text{TSE}}$, and *extended trajectory angular velocity* TRA, $\overline{\text{TRA}}$. Single-trajectory flow measures are attractive because they need minimal information to infer the flow behavior of an underlying field. Obviously, single-trajectory measures cannot be objective in the Euclidean observation space because one may think of a reference system moving with the trajectory, making each trajectory zero [Haller et al., 2021]. Because of this, Haller et al. [2021] considered objectivity in an extended phase space.

Given is a C^2 continuous trajectory $\mathbf{x}(t)$ for $t \in [t_0, t_N]$, its first and second derivatives $\dot{\mathbf{x}}(t)$, $\ddot{\mathbf{x}}(t)$, and a positive constant v_0 accounting for a certain ratio between space and time units to make them non-dimensionalized. Considering $\mathbf{x}(t)$ in an extended phase space gives for the first and second derivative of a trajectory $\underline{\mathbf{x}}(t)$:

$$\underline{\dot{\mathbf{x}}}(t) = \begin{pmatrix} \frac{1}{v_0} \dot{\mathbf{x}}(t) \\ 1 \end{pmatrix}, \quad \underline{\ddot{\mathbf{x}}}(t) = \begin{pmatrix} \frac{1}{v_0} \ddot{\mathbf{x}}(t) \\ 0 \end{pmatrix}. \quad (3)$$

Then a local stretching measure can be defined as

$$\text{tse} = \text{tse}_{\mathbf{x}(t), v_0}(t) = \frac{\underline{\dot{\mathbf{x}}}^T \underline{\ddot{\mathbf{x}}}}{\underline{\dot{\mathbf{x}}}^T \underline{\dot{\mathbf{x}}} + v_0^2} = \frac{\dot{\mathbf{x}}^T \ddot{\mathbf{x}}}{\dot{\mathbf{x}}^T \dot{\mathbf{x}} + v_0^2} \quad (4)$$

from which the Lagrangian measures TSE and $\overline{\text{TSE}}$ are computed by integrating tse along the trajectory:

$$\text{TSE}_{\mathbf{x}(t), v_0}^{t_0, t_N} = \frac{1}{\Delta t} \int_{t_0}^{t_N} \text{tse} dt = \frac{1}{\Delta t} \ln \sqrt{\frac{|\dot{\mathbf{x}}(t_N)|^2 + v_0^2}{|\dot{\mathbf{x}}(t_0)|^2 + v_0^2}} \quad (5)$$

$$\overline{\text{TSE}}_{\mathbf{x}(t), v_0}^{t_0, t_N} = \frac{1}{\Delta t} \int_{t_0}^{t_N} |\text{tse}| dt \approx \frac{1}{\Delta t} \sum_{i=0}^{N-1} \left| \ln \sqrt{\frac{|\dot{\mathbf{x}}(t_{i+1})|^2 + v_0^2}{|\dot{\mathbf{x}}(t_i)|^2 + v_0^2}} \right| \quad (6)$$

with $\Delta t = t_N - t_0$. The discretization in Eq. (6) samples $\mathbf{x}(t)$ at $N + 1$ time steps $t_0 < t_1 < \dots < t_N$.

For defining TRA, the $(n + 1)$ -dimensional matrix function

$$\mathbf{tra} = \mathbf{tra}_{\mathbf{x}(t), v_0}(t) = \frac{\underline{\dot{\mathbf{x}}} \underline{\ddot{\mathbf{x}}}^T - \underline{\ddot{\mathbf{x}}} \underline{\dot{\mathbf{x}}}^T}{\underline{\dot{\mathbf{x}}}^T \underline{\dot{\mathbf{x}}}} \quad (7)$$

can be introduced that describes the local angular velocity. Note that \mathbf{tra} is an anti-symmetric matrix, from which one gets by integration along the trajectory Lagrangian measures

$$\text{TRA}_{\mathbf{x}(t), v_0}^{t_0, t_N} = \frac{1}{\Delta t} \frac{\sqrt{2}}{2} \left| \int_{t_0}^{t_N} \mathbf{tra} dt \right|_{Fr} \quad (8)$$

$$= \frac{1}{\Delta t} \cos^{-1} \frac{\dot{\mathbf{x}}(t_0)^T \dot{\mathbf{x}}(t_N) + v_0^2}{\sqrt{|\dot{\mathbf{x}}(t_0)|^2 + v_0^2} \sqrt{|\dot{\mathbf{x}}(t_N)|^2 + v_0^2}} \quad (9)$$

$$\overline{\text{TRA}}_{\mathbf{x}(t), v_0}^{t_0, t_N} = \frac{1}{\Delta t} \frac{\sqrt{2}}{2} \int_{t_0}^{t_N} |\mathbf{tra}|_{Fr} dt \quad (10)$$

$$\approx \frac{1}{\Delta t} \sum_{i=0}^{N-1} \cos^{-1} \frac{\dot{\mathbf{x}}(t_i)^T \dot{\mathbf{x}}(t_{i+1}) + v_0^2}{\sqrt{|\dot{\mathbf{x}}(t_i)|^2 + v_0^2} \sqrt{|\dot{\mathbf{x}}(t_{i+1})|^2 + v_0^2}} \quad (11)$$

where $\| \cdot \|_{F_T}$ denotes the Frobenius norm of a matrix. Haller et al. [2021] claimed that TSE and $\overline{\text{TSE}}$ are objective in the extended phase space, and that TRA and $\overline{\text{TRA}}$ are quasi-objective in the extended phase space under a certain condition put to the average vorticity in a certain neighborhood of the trajectory.

2.2 TSE, TRA, and underlying velocity fields

We recapitulate the definition of TSE from Haller et al. [2021], keeping their notation as much as possible. We start with a single observed trajectory $\mathbf{x}(t)$ in n -D ($n = 2, 3$) for $t \in [t_0, t_N]$ running from $\mathbf{x}_0 = \mathbf{x}(t_0)$ to $\mathbf{x}_N = \mathbf{x}(t_N)$. Further, we assume that $\mathbf{x}(t)$ is a trajectory (path line) of an underlying unsteady velocity field $\mathbf{v}(\mathbf{x}, t)$, i.e., $\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}(t), t)$ for all $t \in [t_0, t_N]$. Following Haller et al. [2021], \mathbf{v} is transformed into a non-dimensionalized field \mathbf{u} by

$$\mathbf{y} = \frac{\mathbf{x}}{L}, \quad \tau = \tau_0 + \frac{t - t_0}{T}, \quad v_0 = \frac{L}{T} \quad (12)$$

where L, T, v_0 are certain positive constants for a field that need to be determined by additional knowledge about the data. Generally, the scaling factor v_0 is non-zero, i.e., $v_0 \neq 0$. This transformation rephrases $\mathbf{x}(t)$ into the non-dimensionalized trajectory

$$\mathbf{y}(\tau) = \frac{1}{L} \mathbf{x}(t_0 + T(\tau - \tau_0)) \quad (13)$$

running from $\mathbf{y}_0 = \mathbf{y}(\tau_0) = \frac{1}{L} \mathbf{x}_0$ to $\mathbf{y}_N = \mathbf{y}(\tau_N) = \frac{1}{L} \mathbf{x}_N$ with $\tau_N = \tau_0 + \frac{t_N - t_0}{T}$. Further, it gives the non-dimensionalized vector field

$$\mathbf{u}(\mathbf{y}, \tau) = \frac{1}{v_0} \mathbf{v}(L\mathbf{y}, t_0 + T(\tau - \tau_0)). \quad (14)$$

Note that (14) contains a correction of a missing term $\frac{1}{v_0}$ in formula (26) in [Haller et al., 2021]. The error in formula (26) in [Haller et al., 2021] can be seen in the following way: suppose \mathbf{v} is a constant vector field, i.e., $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_c$. Then formula (26) in [Haller et al., 2021] would give $\mathbf{u}(\mathbf{y}, \tau) = \mathbf{v}_c$ no matter how v_0 is chosen. This would contradict to the formula before (33) in [Haller et al., 2021].

Following Haller et al. [2021] further, an extended phase space $\mathbf{Y} = \begin{pmatrix} \mathbf{y} \\ z \end{pmatrix}$ is introduced by adding the additional time dimension z . Transformation of $\mathbf{y}(\tau)$ and $\mathbf{u}(\mathbf{y}, \tau)$ into this extended phase space gives

$$\mathbf{Y}(\tau) = \begin{pmatrix} \mathbf{y}(\tau) \\ \tau \end{pmatrix}, \quad \mathbf{U}(\mathbf{Y}) = \begin{pmatrix} \mathbf{u}(\mathbf{y}, z) \\ 1 \end{pmatrix} \quad (15)$$

where $\mathbf{Y}(\tau)$ is the trajectory in the extended phase space running from $\mathbf{Y}_0 = \mathbf{Y}(\tau_0) = \begin{pmatrix} \mathbf{y}_0 \\ \tau_0 \end{pmatrix}$ to $\mathbf{Y}_N = \mathbf{Y}(\tau_N) = \begin{pmatrix} \mathbf{y}_N \\ \tau_N \end{pmatrix}$, and $\mathbf{U}(\mathbf{Y})$ is the underlying vector field. The tangent vector of $\mathbf{Y}(\tau)$ is

$$\mathbf{Y}'(\tau) = \frac{d\mathbf{Y}}{d\tau} = \begin{pmatrix} \mathbf{y}'(\tau) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{v_0} \dot{\mathbf{x}}(t_0 + T(\tau - \tau_0)) \\ 1 \end{pmatrix}. \quad (16)$$

Note that $\mathbf{U}(\mathbf{Y})$ is an autonomous dynamical system now: \mathbf{U} is a steady velocity field in the extended phase space. Then Haller et al. [2021] defines TSE and TRA as

$$\text{TSE}_{t_0}^{t_N}(\mathbf{x}_0, v_0) = \frac{1}{\Delta t} \ln \frac{|\mathbf{Y}'(\tau_N)|}{|\mathbf{Y}'(\tau_0)|} = \frac{1}{\Delta t} \ln \frac{|\mathbf{U}(\mathbf{Y}_N)|}{|\mathbf{U}(\mathbf{Y}_0)|} \quad (17)$$

$$\text{TRA}_{t_0}^{t_N}(\mathbf{x}_0, v_0) = \frac{1}{\Delta t} \cos^{-1} \frac{\mathbf{Y}'(\tau_0)^T \mathbf{Y}'(\tau_N)}{|\mathbf{Y}'(\tau_0)| |\mathbf{Y}'(\tau_N)|} \quad (18)$$

$$= \frac{1}{\Delta t} \cos^{-1} \frac{\mathbf{U}(\mathbf{Y}_0)^T \mathbf{U}(\mathbf{Y}_N)}{|\mathbf{U}(\mathbf{Y}_0)| |\mathbf{U}(\mathbf{Y}_N)|} \quad (19)$$

where $\Delta t = t_N - t_0$, (17) is identical to the right-hand side of (5), and (18) is identical to the right-hand side of (9). To show objectivity of TSE in the extended phase space, one has to prove that TSE is invariant under observation in any moving Euclidean reference system in the extended phase space. Analogous to Eq. (1), such moving reference system is defined by

$$\mathbf{Y} = \mathbf{Q}(\tau) \tilde{\mathbf{Y}} + \mathbf{B}(\tau), \quad \mathbf{Q}(\tau) = \begin{pmatrix} \mathbf{Q}(\tau) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}, \quad \mathbf{B}(\tau) = \begin{pmatrix} \mathbf{b}(\tau) \\ 0 \end{pmatrix} \quad (20)$$

with $\mathbf{Q}(\tau) \in SO(n)$ being a rotation matrix, and $\mathbf{0}$ being the zero-vector. The observed trajectory $\tilde{\mathbf{Y}}(\tau)$ and the underlying velocity field $\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}, \tau)$ in the new moving reference system are

$$\tilde{\mathbf{Y}}(\tau) = \mathbf{Q}^T(\tau)(\mathbf{Y}(\tau) - \mathbf{B}(\tau)) \quad (21)$$

$$\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}, \tau) = \mathbf{Q}^T(\tau) \left(\mathbf{U} \left(\mathbf{Q}(\tau)\tilde{\mathbf{Y}} + \mathbf{B}(\tau) \right) - \dot{\mathbf{Q}}(\tau)\tilde{\mathbf{Y}} - \dot{\mathbf{B}}(\tau) \right) \quad (22)$$

where the new trajectory $\tilde{\mathbf{Y}}(\tau)$ runs from $\tilde{\mathbf{Y}}_0 = \tilde{\mathbf{Y}}(\tau_0)$ to $\tilde{\mathbf{Y}}_N = \tilde{\mathbf{Y}}(\tau_N)$. Then, TSE in the moving reference system is

$$\widetilde{\text{TSE}}_{t_0}^{t_N}(\mathbf{x}_0, v_0) = \frac{1}{\Delta t} \ln \frac{|\tilde{\mathbf{Y}}'(\tau_N)|}{|\tilde{\mathbf{Y}}'(\tau_0)|} = \frac{1}{\Delta t} \ln \frac{|\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}_N, \tau_N)|}{|\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}_0, \tau_0)|} \quad (23)$$

$$\widetilde{\text{TRA}}_{t_0}^{t_N}(\mathbf{x}_0, v_0) = \frac{1}{\Delta t} \cos^{-1} \frac{\tilde{\mathbf{Y}}'(\tau_0)^T \tilde{\mathbf{Y}}'(\tau_N)}{|\tilde{\mathbf{Y}}'(\tau_0)| |\tilde{\mathbf{Y}}'(\tau_N)|} \quad (24)$$

To show objectivity of TSE in the extended phase space, one has to prove $\widetilde{\text{TSE}} = \text{TSE}$ for any moving reference frame, as given by Eq. (20). To show quasi-objectivity of TRA under averaged-vorticity condition, one has to prove $\widetilde{\text{TRA}} = \text{TRA}$ for all reference frames (20) in which the averaged-vorticity condition is fulfilled.

3 The first counter-example

We show the non-objectivity of TSE in the extended phase space by a simple counter-example. We set the 2D observed trajectory $\mathbf{x}(t)$ and the underlying velocity field $\mathbf{v}(\mathbf{x}, t)$ as

$$\mathbf{x}(t) = \begin{pmatrix} e^t - 1 \\ t(t+1) \end{pmatrix}, \quad \mathbf{v}(\mathbf{x}, t) = \begin{pmatrix} x+1 \\ 2t+1 \end{pmatrix} \quad (25)$$

for $t \in [t_0, t_N] = [0, 1]$ and $\mathbf{x} = (x, y)^T$. To calculate TSE as in Eq. (5), we only need information at time t_0 and t_N . This gives

$$\mathbf{x}_0 = (0, 0)^T, \quad \mathbf{x}_N = (e-1, 2)^T \quad (26)$$

$$\dot{\mathbf{x}}(t_0) = \mathbf{v}(\mathbf{x}_0, t_0) = (1, 1)^T, \quad \dot{\mathbf{x}}(t_N) = \mathbf{v}(\mathbf{x}_N, t_N) = (e, 3)^T. \quad (27)$$

For the non-dimensionalization transformation, we set $\tau_0 = 0$, resulting in $\tau_N = \frac{1}{T}$. This gives with Eqs. (13) and (14)

$$\mathbf{y}(\tau) = \frac{1}{L} \begin{pmatrix} e^{T\tau} - 1 \\ T\tau(T\tau + 1) \end{pmatrix}, \quad \mathbf{u}(\mathbf{y}, \tau) = \frac{1}{v_0} \begin{pmatrix} L\bar{x} + 1 \\ 2T\tau + 1 \end{pmatrix} \quad (28)$$

with $\mathbf{y} = (\bar{x}, \bar{y})^T$, and therefore we obtain at τ_0 and τ_N

$$\mathbf{y}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_N = \frac{1}{L} \begin{pmatrix} e-1 \\ 2 \end{pmatrix} \quad (29)$$

$$\mathbf{u}(\mathbf{y}_0, \tau_0) = \frac{1}{v_0} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}(\mathbf{y}_N, \tau_N) = \frac{1}{v_0} \begin{pmatrix} e \\ 3 \end{pmatrix}. \quad (30)$$

Transforming to the extended phase space

$$\mathbf{Y} = (\bar{x}, \bar{y}, z)^T \quad (31)$$

using Eq. (15) gives

$$\mathbf{Y}(\tau) = \begin{pmatrix} \frac{1}{L}(e^{T\tau} - 1) \\ \frac{1}{L}T\tau(T\tau + 1) \\ \tau \end{pmatrix}, \quad \mathbf{U}(\mathbf{Y}) = \begin{pmatrix} \frac{1}{v_0}(L\bar{x} + 1) \\ \frac{1}{v_0}(2Tz + 1) \\ 1 \end{pmatrix} \quad (32)$$

with the following position and tangent at the curve end points

$$\mathbf{Y}_0 = (0, 0, 0)^T, \quad \mathbf{Y}_N = \left(\frac{e-1}{L}, \frac{2}{L}, \frac{1}{T} \right)^T \quad (33)$$

$$\mathbf{U}(\mathbf{Y}_0) = \left(\frac{1}{v_0}, \frac{1}{v_0}, 1 \right)^T, \quad \mathbf{U}(\mathbf{Y}_N) = \left(\frac{e}{v_0}, \frac{3}{v_0}, 1 \right)^T. \quad (34)$$

Inserting into Eqs. (17) and (18), this results in TSE and TRA:

$$\text{TSE} = \ln \sqrt{\frac{e^2 + 9 + v_0^2}{2 + v_0^2}}, \quad \text{TRA} = \cos^{-1} \frac{e + 3 + v_0^2}{\sqrt{2 + v_0^2} \sqrt{e^2 + 9 + v_0^2}}. \quad (35)$$

For our counterexample, it is sufficient to choose a particular moving Euclidean reference system (20) by

$$\mathbf{Q}(\tau) = \mathbf{I} \quad , \quad \mathbf{B}(\tau) = \tau \begin{pmatrix} \mathbf{b}_c \\ 0 \end{pmatrix} \quad (36)$$

where \mathbf{I} is the identity matrix and $\mathbf{b}_c = (x_c, y_c)^T$ is a constant 2D vector. For this particular reference system, we get by (21), (22):

$$\tilde{\mathbf{Y}}(\tau) = \mathbf{Y}(\tau) - \tau \begin{pmatrix} \mathbf{b}_c \\ 0 \end{pmatrix} \quad (37)$$

$$\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}, \tau) = \mathbf{U} \left(\mathbf{Y} + \tau \begin{pmatrix} \mathbf{b}_c \\ 0 \end{pmatrix} \right) - \begin{pmatrix} \mathbf{b}_c \\ 0 \end{pmatrix}. \quad (38)$$

This gives the following trajectory end points and tangents:

$$\tilde{\mathbf{Y}}_0 = (0, 0, 0)^T \quad , \quad \tilde{\mathbf{Y}}_N = \left(\frac{e-1}{L} - \frac{x_c}{T}, \frac{2}{L} - \frac{y_c}{T}, \frac{1}{T} \right)^T \quad (39)$$

$$\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}_0, \tau_0) = \tilde{\mathbf{Y}}'(\tau_0) = \left(\frac{1}{v_0} - x_c, \frac{1}{v_0} - y_c, 1 \right)^T \quad (40)$$

$$\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}_N, \tau_N) = \tilde{\mathbf{Y}}'(\tau_N) = \left(\frac{e}{v_0} - x_c, \frac{3}{v_0} - y_c, 1 \right)^T \quad (41)$$

and finally by inserting into Eq. (23), we get $\widetilde{\text{TSE}}$:

$$\widetilde{\text{TSE}} = \ln \sqrt{\frac{(e - v_0 x_c)^2 + (3 - v_0 y_c)^2 + v_0^2}{(1 - v_0 x_c)^2 + (1 - v_0 y_c)^2 + v_0^2}} \quad (42)$$

Analogously, $\widetilde{\text{TRA}}$ follows by inserting (40)–(41) into (24). Since there is no positive constant v_0 , cf. (12), that makes TSE in (35) and $\widetilde{\text{TSE}}$ in (42) identical for any $\mathbf{b}_c = (x_c, y_c)^T$, non-objectivity of TSE in the extended phase space is shown. Since in our example both $\mathbf{U}(\mathbf{Y})$ and $\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}, \tau)$ have zero vorticity, the average-vorticity condition in Haller et al. [Haller et al., 2021] is trivially fulfilled. Thus, the difference of TRA and $\widetilde{\text{TRA}}$ gives that TRA is not quasi-objective in the extended phase under the average-vorticity condition.

Where is the error?

Haller et al. [2021] considered a non-zero vector ξ_0 at (\mathbf{x}_0, t_0) that is advected with \mathbf{v} along $\mathbf{x}(t)$, resulting in

$$\dot{\xi}(t) = \nabla \mathbf{v}(\mathbf{x}(t), t) \xi(t) \quad , \quad \xi(t_0) = \xi_0. \quad (43)$$

Then, $\xi(t)$ is observed under a moving reference system (1). Objectivity of ξ is deduced from (43), (1):

$$\tilde{\xi}(t) = \mathbf{Q}^T(t) \xi(t) \quad (44)$$

where $\tilde{\xi}$ is the observation of ξ under the moving reference system (1). From (44) follows the objectivity of $\frac{1}{\Delta t} \ln \frac{|\xi(t_N)|}{|\xi_0|}$. We note that (44) follows from (43) and (1) only if another implicit assumption holds: objectivity of the seeding vector ξ_0 , i.e., $\tilde{\xi}_0 = \mathbf{Q}^T(t_0) \xi_0$.

The approach of Haller et al. [2021] is to set $\xi_0 = \mathbf{v}_0 = \mathbf{v}(\mathbf{x}_0, t_0)$. With this, additional conditions are necessary to ensure

$$\dot{\mathbf{v}}(t) = \nabla \mathbf{v}(\mathbf{x}(t), t) \mathbf{v}(\mathbf{x}, t) \quad (45)$$

$$\tilde{\mathbf{v}}(\tilde{\mathbf{x}}, t) = \mathbf{Q}^T(t) \mathbf{v}(\mathbf{x}, t) \quad (46)$$

where (45) corresponds to (43) and (46) corresponds to (44). To ensure (45), Haller et al. [2021] introduced the condition

$$(A1) \quad \partial_t \mathbf{v}(\mathbf{x}, t) = \mathbf{0}$$

in the current observation frame. However, condition (A1) does not ensure (46) because $\xi_0 = \mathbf{v}_0$ is not objective. Since the observation of \mathbf{v} under the moving reference system (1) is [Haller, 2021]

$$\tilde{\mathbf{v}}(\tilde{\mathbf{x}}, t) = \mathbf{Q}^T(t) \left(\mathbf{v}(\mathbf{x}, t) - \dot{\mathbf{Q}}(t) \tilde{\mathbf{x}} - \dot{\mathbf{b}}(t) \right), \quad (47)$$

Eq. (46) is in general only fulfilled for $\dot{\mathbf{Q}} = \mathbf{0}$, $\dot{\mathbf{b}} = \mathbf{0}$, i.e., the reference frame is not moving but static, resulting in demanding that $\tilde{\mathbf{v}}(\tilde{\mathbf{x}}, t)$ is steady. This means that the condition for the quasi-objectivity of TSE is the steadiness of both \mathbf{v} and $\tilde{\mathbf{v}}$ in all considered reference frames. We remark that this is a rather strong condition for quasi-objectivity: it excludes the consideration of all moving reference frames.

The transformation to the extended reference system transforms \mathbf{v} to the steady vector field \mathbf{U} , making the condition (A1) for (45) in the extended reference frame obsolete. However, the observation $\tilde{\mathbf{U}}$ of \mathbf{U} under a moving reference system (20) is *not* a steady vector field anymore, as shown in (22). This means that

$$\tilde{\mathbf{U}}(\tilde{\mathbf{Y}}, \tau) = \mathcal{Q}(\tau) \mathbf{U}(\mathbf{Y}) \quad (48)$$

does not hold in general but only for particular steady reference frames. Because of this, TSE is in the extended phase space not objective but only quasi-objective under restriction to a static reference system.

Summary: The error was to assume that the observation of an autonomous system (steady vector field) in the extended phase space under a moving reference frame remains an autonomous system.

Remarks: A similar argumentation gives that $\overline{\text{TSE}}$ is not objective in the extended phase space, and that $\overline{\text{TRA}}$ is not quasi-objective in the extended phase space under the averaged-vorticity-based condition. TSE, $\overline{\text{TSE}}$, TRA and $\overline{\text{TRA}}$ are not even Galilean invariant because the moving reference system (36) in the counterexample was performing a Galilean transformation.

4 The second counter-example

In an erratum, Haller et al. [2022] give up the idea of considering extended phase space and non-dimensionalization, and introduce a new condition

$$(A3) \quad |\partial_t \mathbf{v}(\mathbf{x}(t), t)| \ll |\ddot{\mathbf{x}}(t)|. \quad (49)$$

Based on this, Theorem 3 in Haller et al. [2021] is reformulated. Unfortunately, the reformulated theorem is still incorrect. To show this, we consider a second counter-example. We consider the 3D trajectory and the underlying velocity field

$$\mathbf{x}(t) = \begin{pmatrix} e^t \\ -t \\ 0 \end{pmatrix}, \quad \mathbf{v}(\mathbf{x}, t) = \begin{pmatrix} x \\ -1 \\ 0 \end{pmatrix} \quad (50)$$

with $\mathbf{x} = (x, y, z)^T$ and t running from $t_0 = 0$ to $t_N = 1$. Observing \mathbf{x} , \mathbf{v} under a reference system (1) with

$$\mathbf{Q}(t) = \mathbf{I}, \quad \mathbf{b}(t) = (0, t, 0)^T \quad (51)$$

gives

$$\tilde{\mathbf{x}}(t) = \begin{pmatrix} e^t \\ -2t \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, t) = \begin{pmatrix} \tilde{x} \\ -2 \\ 0 \end{pmatrix} \quad (52)$$

with $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{z})^T$. We note that

$$\partial_t \mathbf{v}(\mathbf{x}, t) = \partial_t \tilde{\mathbf{v}}(\tilde{\mathbf{x}}, t) = \boldsymbol{\omega}(\mathbf{x}, t) = \tilde{\boldsymbol{\omega}}(\tilde{\mathbf{x}}, t) = \mathbf{0} \quad (53)$$

$$\ddot{\mathbf{x}}(t) = \tilde{\ddot{\mathbf{x}}}(t) = (e^t, 0, 0)^T \quad (54)$$

where $\boldsymbol{\omega}$, $\tilde{\boldsymbol{\omega}}$ are the vorticity of \mathbf{v} , $\tilde{\mathbf{v}}$, respectively. This means that all conditions (A1),(A2) (from Haller et al. [2021]) and (A3) (from Haller et al. [2022]) are fulfilled both in the original frame in (50) and in the particular moving reference frame in (52). Computing TSE, $\overline{\text{TSE}}$, TRA, $\overline{\text{TRA}}$ from (50), and computing $\widetilde{\text{TSE}}$, $\widetilde{\overline{\text{TSE}}}$, $\widetilde{\text{TRA}}$, $\widetilde{\overline{\text{TRA}}}$ from (52) gives

$$\text{TSE} = \overline{\text{TSE}} = \frac{\ln(e^2 + 1) - \ln(2)}{2} \approx 0.71689 \quad (55)$$

$$\widetilde{\text{TSE}} = \widetilde{\overline{\text{TSE}}} = \frac{\ln(e^2 + 4) - \ln(5)}{2} \approx 0.41161 \quad (56)$$

$$\text{TRA} = \overline{\text{TRA}} = \tan^{-1}(e) - \frac{\pi}{4} \approx 0.43288 \quad (57)$$

$$\widetilde{\text{TRA}} = \widetilde{\overline{\text{TRA}}} = \tan^{-1}(e/2) - \tan^{-1}(1/2) \approx 0.47282 \quad (58)$$

For TSE being quasi-objective under condition (A3) following Haller et al. [2021], there must be an objective scalar \mathcal{P} such that

$$\text{TSE} \approx \mathcal{P} \quad , \quad \widetilde{\text{TSE}} \approx \widetilde{\mathcal{P}} \quad (59)$$

where $\widetilde{\mathcal{P}}$ is the observation of \mathcal{P} under reference frame (51) and “the accuracy of the approximation indicated by the symbol \approx depends on the extend to which the conditions (A3) is satisfied” Haller et al. [2021]. Note that here assumption (A3) is completely satisfied, since the value $|\partial_t \mathbf{v}(\mathbf{x}(t), t)|/|\ddot{\mathbf{x}}(t)|$ as considered in Figure 1 of Haller et al. [2022] is constant zero in both reference frames, see Eq. (53)–(54). Since $\mathcal{P} = \widetilde{\mathcal{P}}$ due to the demanded objectivity of \mathcal{P} and since assumption (A3) is completely satisfied, Eqs. (55) and (56) give that (59) cannot be fulfilled since TSE and $\widetilde{\text{TSE}}$ are significantly different. Hence, TSE is not quasi-objective under condition (A3).

Eqs. (55)–(58) give that similar statements hold for all measures TSE, $\overline{\text{TSE}}$, TRA, $\overline{\text{TRA}}$: none of them is quasi-objective under any combination of conditions (A1), (A2), (A3). This example and the example from Section 3 show that all theorems in both Haller et al. [2021] and Haller et al. [2022] are incorrect.

References

- George Haller, Nikolas Aksamit, and Alex P. Encinas Bartos. Quasi-objective coherent structure diagnostics from single trajectories. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 31(4):043131, 2021. doi:10.1063/5.0044151. URL <https://doi.org/10.1063/5.0044151>.
- George Haller, Nikolas Aksamit, and Alex P. Encinas Bartos. Erratum: "Quasi-objective coherent structure diagnostics from single trajectories" [Chaos 31, 043131 (2021)]. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 32(5):059901, 2022. doi:10.1063/5.0090124. URL <https://doi.org/10.1063/5.0090124>.
- Alex P. Encinas-Bartos, Nikolas O. Aksamit, and George Haller. Quasi-objective eddy visualization from sparse drifter data. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 32(11):113143, 2022. doi:10.1063/5.0099859. URL <https://doi.org/10.1063/5.0099859>.
- Nikolas O. Aksamit and George Haller. Objective momentum barriers in wall turbulence. *Journal of Fluid Mechanics*, 941:A3, 2022. doi:10.1017/jfm.2022.316.
- Holger Theisel, Anke Friederici, and Tobias Günther. Objective flow measures based on few trajectories, 2022. URL <https://arxiv.org/abs/2202.09566>.
- Clifford Truesdell and Walter Noll. *The Nonlinear Field Theories of Mechanics*. Springer, 1965.
- George Haller. Can vortex criteria be objectivized? *Journal of Fluid Mechanics*, 508:A25, 2021. doi:10.1017/jfm.2020.937.