# Appendix: Trajectory Vorticity - Computation and Visualization of Rotational Trajectory Behavior in an Objective Way

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## Appendix

What follows is the proof that $\mathbf{trv}$ as defined in (27)–(38) is objective. We consider the observation of the trajectories in a moving reference system performing a Euclidean transformation of the form as in Eq. (1). We denote the observed measures from (27)–(38) with a tilde. This gives for the observations of $\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}$ in the new reference system:

$$\widetilde{\mathbf{X}} = \boldsymbol{Q}^{\mathrm{T}}(\mathbf{X} - \mathbf{B}) \tag{1}$$
$$\dot{\widetilde{\mathbf{X}}} = \dot{\boldsymbol{Q}}^{\mathrm{T}}(\mathbf{X} - \mathbf{B}) + \boldsymbol{Q}^{\mathrm{T}}(\dot{\mathbf{X}} - \dot{\mathbf{B}}) \tag{2}$$
$$\ddot{\widetilde{\mathbf{X}}} = \ddot{\boldsymbol{Q}}^{\mathrm{T}}(\mathbf{X} - \mathbf{B}) + 2\dot{\boldsymbol{Q}}^{\mathrm{T}}(\dot{\mathbf{X}} - \dot{\mathbf{B}}) + \boldsymbol{Q}^{\mathrm{T}}(\ddot{\mathbf{X}} - \ddot{\mathbf{B}}) \tag{3}$$

where $\mathbf{B}, \dot{\mathbf{B}}, \ddot{\mathbf{B}}$ are $((n+1) \times m)$ matrices and $\boldsymbol{Q}, \dot{\boldsymbol{Q}}, \ddot{\boldsymbol{Q}}$ are $((n+1) \times (n+1))$ matrices defined as

$$\mathbf{B} = \begin{pmatrix} \mathbf{b} & \dots & \mathbf{b} \\ 0 & \dots & 0 \end{pmatrix}, \quad \dot{\mathbf{B}} = \begin{pmatrix} \dot{\mathbf{b}} & \dots & \dot{\mathbf{b}} \\ 0 & \dots & 0 \end{pmatrix}, \quad \ddot{\mathbf{B}} = \begin{pmatrix} \ddot{\mathbf{b}} & \dots & \ddot{\mathbf{b}} \\ 0 & \dots & 0 \end{pmatrix}$$
$$\boldsymbol{Q} = \begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{pmatrix}, \quad \dot{\boldsymbol{Q}} = \begin{pmatrix} \dot{\mathbf{Q}} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}, \quad \ddot{\boldsymbol{Q}} = \begin{pmatrix} \ddot{\mathbf{Q}} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}.$$

Eqs. (31) and (1) give

$$\widetilde{\mathbf{X}}^{-1} = \mathbf{X}^{-1} \begin{pmatrix} \mathbf{I} & \mathbf{b} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{pmatrix} \boldsymbol{Q} \tag{4}$$
$$\widetilde{\mathbf{H}} = \boldsymbol{Q}^{\mathrm{T}} \mathbf{H} \boldsymbol{Q} + \dot{\boldsymbol{Q}}^{\mathrm{T}} \boldsymbol{Q} + \left( \mathbf{0} \;,\; \boldsymbol{Q}^{\mathrm{T}} \left( \dot{\mathbf{X}} \mathbf{X}^{-1} \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} - \begin{pmatrix} \dot{\mathbf{b}} \\ 0 \end{pmatrix} \right) \right) \tag{5}$$

and from Eqs. (32) and (5) follows

$$\begin{aligned} \widetilde{\mathbf{J}} &= \mathbf{Q}^{\mathrm{T}} \mathbf{J} \mathbf{Q} + \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{Q} \\ \dot{\widetilde{\mathbf{J}}} &= \mathbf{Q}^{\mathrm{T}} \dot{\mathbf{J}} \mathbf{Q} + \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{J} \mathbf{Q} + \mathbf{Q}^{\mathrm{T}} \mathbf{J} \dot{\mathbf{Q}} + \dot{\mathbf{Q}}^{\mathrm{T}} \dot{\mathbf{Q}} + \ddot{\mathbf{Q}}^{\mathrm{T}} \mathbf{Q}. \end{aligned}$$

Since $\dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{Q}$ and $\dot{\mathbf{Q}}^{\mathrm{T}} \dot{\mathbf{Q}} + \ddot{\mathbf{Q}}^{\mathrm{T}} \mathbf{Q}$ are anti-symmetric, we get

$$\widetilde{\mathbf{W}} = \mathbf{R}^{\mathrm{T}} \mathbf{W} \mathbf{Q} + \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{Q} \tag{6}$$
$$\widetilde{\mathbf{S}} = \mathbf{Q}^{\mathrm{T}} \mathbf{S} \mathbf{Q} \tag{7}$$
$$\dot{\widetilde{\mathbf{S}}} = \mathbf{Q}^{\mathrm{T}} \dot{\mathbf{S}} \mathbf{Q} + \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{S} \mathbf{Q} + \mathbf{Q}^{\mathrm{T}} \mathbf{S} \dot{\mathbf{Q}}. \tag{8}$$

Eq. (7) gives: if $\mathbf{e}$ is an eigenvector of $\mathbf{S}$, then $\mathbf{Q}^{\mathrm{T}} \mathbf{e}$ is an eigenvector of $\widetilde{\mathbf{S}}$. From this follows

$$\widetilde{\mathbf{E}} = \mathbf{Q}^{\mathrm{T}} \mathbf{E} \tag{9}$$

which gives

$$\widetilde{\overline{\mathbf{S}}} = \widetilde{\mathbf{E}}^{\mathrm{T}} \widetilde{\mathbf{S}} \widetilde{\mathbf{E}} = (\mathbf{E}^{\mathrm{T}} \mathbf{Q}) (\mathbf{Q}^{\mathrm{T}} \mathbf{S} \mathbf{Q}) (\mathbf{Q}^{\mathrm{T}} \mathbf{E}) = \overline{\mathbf{S}} \tag{10}$$
$$\begin{aligned} \widetilde{\dot{\overline{\mathbf{S}}}} &= \widetilde{\mathbf{E}}^{\mathrm{T}} \dot{\widetilde{\mathbf{S}}} \widetilde{\mathbf{E}} && (11) \\ &= \dot{\overline{\mathbf{S}}} + \mathbf{E}^{\mathrm{T}} (\mathbf{Q} \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{S} + \mathbf{S} \dot{\mathbf{Q}} \mathbf{Q}^{\mathrm{T}}) \mathbf{E} && (12) \\ &= \dot{\overline{\mathbf{S}}} + (\mathbf{E}^{\mathrm{T}} \mathbf{Q} \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{E}) \overline{\mathbf{S}} + \overline{\mathbf{S}} (\mathbf{E}^{\mathrm{T}} \dot{\mathbf{Q}} \mathbf{Q}^{\mathrm{T}} \mathbf{E}) && (13) \\ &= \dot{\overline{\mathbf{S}}} + (\mathbf{E}^{\mathrm{T}} \mathbf{Q} \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{E}) \overline{\mathbf{S}} - \overline{\mathbf{S}} (\mathbf{E}^{\mathrm{T}} \mathbf{Q} \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{E}). && (14) \end{aligned}$$

Eqs. (10) and (14), $\overline{\mathbf{S}}$ being a diagonal matrix, and $(\mathbf{E}^{\mathrm{T}} \mathbf{Q} \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{E})$ being anti-symmetric gives

$$\widetilde{\overline{\mathbf{W}}}_s = \overline{\mathbf{W}}_s + \mathbf{E}^{\mathrm{T}} \mathbf{Q} \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{E}. \tag{15}$$

Then, Eqs. (38), (9), and (15) give

$$\widetilde{\mathbf{W}}_s = \widetilde{\mathbf{E}} \, \widetilde{\overline{\mathbf{W}}}_s \, \widetilde{\mathbf{E}}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}} \mathbf{W}_s \mathbf{Q} + \dot{\mathbf{Q}}^{\mathrm{T}} \mathbf{Q}. \tag{16}$$

Finally, Eqs. (6) and (16) give $\widetilde{\mathbf{trv}} = \widetilde{\mathbf{W}} - \widetilde{\mathbf{W}}_s = \mathbf{Q}^{\mathrm{T}} \, \mathbf{trv} \, \mathbf{Q}$ which proves the theorem.