Visualizing the Curvature of Unsteady 2D Flow Fields

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Abstract. The treatment of stream lines, path lines and streak lines is a powerful method for visualizing the behavior of unsteady (i.e. time dependent) flow fields. Although a closed parametric form of these lines does not exist, we show how to compute their curvatures. The visualization of the curvatures of these curves shows several aspects of the flow, such as location and movement of turbulent areas. The technique works without applying any numerical integrations.
Keywords: Flow visualization, stream lines, streak lines, path lines, time lines, curvature.

1 Introduction

The visualization of flow data has become one of the research topics in scientific visualization. Several techniques for visualizing the flow of fluids and gases have been developed.

One of the most important approaches for visualizing steady (i.e., time independent) flows is the treatment of stream lines. A stream line can be considered as the path of a massless particle in the flow. Stream lines have interesting visualization properties: Visualizing an appropriate number of them gives an impression of the behavior of the entire flow. Unfortunately, in general stream lines cannot be computed in a direct way but only as the numerical solution of partial differential equations.

Several approaches for integrating and visualizing stream lines can be found in [1], [2], [3], [7] and [9]. In [4] and [5], topological concepts of vector fields are used for visualizing the flow. The critical points of the vector field are detected and classified. These points are connected by particular tangent curves, called separation curves. Unfortunately, the classification of the critical points works only for first order approximations of the flow. In [6] local properties of the flow (divergence of the vector field, curvature of the tangent curves...) are computed and visualized in an icon, called a local probe. [8] can be considered as a first approach to handle vector fields with higher order topologies. Here the vector fields are described in terms of Clifford algebras.

In [10], the curvatures of stream lines are used as a global visualization method: for every point of the domain of a 2D vector field there is only one stream line through it (except for critical points). In [10] it has been shown that the curvature of a stream line can be computed in a simple closed form even if a closed form for the stream lines themselves do not exist. The curvatures were computed for every point of the domain and color coded using a continuous color coding map with the following properties: red color means positive curvature, green color means negative curvature, the higher the curvature the lighter the color is. In fact, a zero curvature gives black; if the curvature tends to plus (minus) infinity, the red (green) color becomes white.

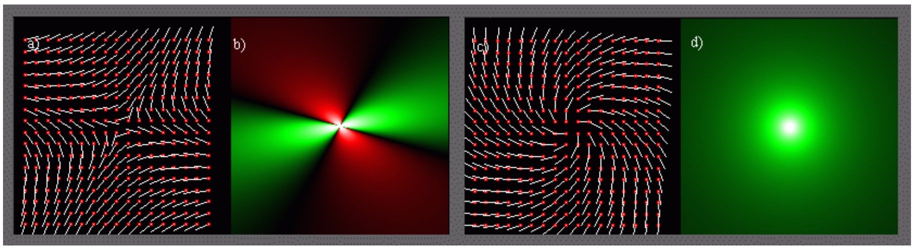

Fig. 1. Examples of stream line curvature for steady vector fields. a) arrow plot of a saddle point, b) curvature visualization of a), c) arrow plot of repelling focus, d) curvature visualization of c).

Figure 1 a) and b) show an example of a vector field with a critical point - a saddle point. Figure 1a) is the arrow plot, figure 1b) shows the curvature visualization of a). The critical point can be seen clearly as a highlight. Figures 1 c) and d) show another example: a vector field with a repelling focus.

In general, critical points can be detected as highlights in the curvature visualization. The pictures obtained by visualizing the stream line curvature give a reasonably good impression of the flow. Since in turbulent areas the flow directions change frequently, these areas generally have high curvatures of the stream lines and can therefore be clearly recognized as highlights. The curvature of stream lines can be computed without applying any numerical integration methods. So there is no risk of destroying the topology of the vector field describing the flow.

In [11] a similar approach was used to visualize the curvature characteristic curves on free-form-surfaces. These curves (lines of curvature, reflection lines ...) can be considered as tangent curves of certain vector fields and therefore be visualized using the methods described in [10].

In this paper, we want to extend the approach of [10] to unsteady 2D flows. Here we have four kinds of characteristic lines: stream lines, path lines, streak lines and time lines. We show how to compute the curvature of stream lines, path lines and streak lines and apply it as a visualization technique for unsteady 2D flows.

Section 2 gives the theoretical background for computing the curvature of the characteristic lines. Sections 3-6 treat stream lines, path lines, streak lines and time lines of unsteady 2D flows. In section 7 the technique is discussed at an example data set.

2 Theoretical Background

A 2D steady flow is usually described as a 2D vector field $V(x, y) = (u(x, y), v(x, y))^T$. Then the stream line of the flow correlates with the *tangent curves* of V. A curve $L \subseteq \mathbb{E}^2$ is called a tangent curve of the vector field V if the following condition is satisfied: For all points $P \in L$, the tangent vector of the curve in the point P has the same direction as the vector $V(P)$.
For every point $P \in \mathbb{E}^2$ there is one and only one tangent curve through it (except for critical points, i.e. points with $\|V\| = 0$).

From the definition of tangent curves we know their tangent vector for every point of the domain: $\dot{\mathbf{x}}(x, y) = V(x, y)$. Applying the chain rule, we obtain the second derivative vector for every point of the domain: $\ddot{\mathbf{x}}(x, y) = (u \cdot V_x + v \cdot V_y)(x, y)$. Then we can compute the (signed) curvature of the tangent curve through (x, y): $\kappa(x, y) = \frac{\|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}\|}{\|\dot{\mathbf{x}}\|^3}(x, y)$. See [10] for more details on the curvature of tangent curves.

An unsteady 2D flow can be described as a 3D vector field

$$V(x, y, t) = \begin{pmatrix} u(x, y, t) \\ v(x, y, t) \\ a(x, y, t) \end{pmatrix} \tag{1}$$

where

$$a(x, y, t) \equiv 1 \quad , \quad a_x = a_y = a_t \equiv 0. \tag{2}$$

The auxiliary dimension $a(x, y, t)$ can be interpreted as the time component of the flow. Since time passes at a constant rate, we have $a(x, y, t) \equiv 1$. V has a critical point iff $u^2 + v^2 = 0$. In general, critical points of V denote the turbulent areas of the flow. In the following, V stands only for an unsteady flow field described by (1).

Projecting V into the planes $t = \text{const}$, we obtain another description of an unsteady 2D flow:

$$V_p(x, y, t) = \begin{pmatrix} u(x, y, t) \\ v(x, y, t) \end{pmatrix} . \tag{3}$$

The introduction of both V and V_p for describing unsteady flow fields has technical reasons: Both descriptions are useful for describing stream lines, path lines, streak lines and time lines.

3 Stream Lines

Stream lines are the tangent curves of V_p. For every time and every location there is one and only one stream line through it (except for critical points).

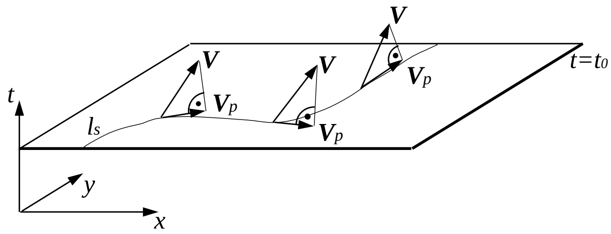

Fig. 2. Stream line l_s of an unsteady flow

Figure 2 shows the computation of the stream lines for the time $t = t_0$. We consider the tangent curves of V_p at this time. We obtain the curvature of the stream lines by computing first and second derivative vector of the stream line for every point of the domain of V_p:

$$\dot{\mathbf{x}}_{stream}(x,y,t) = V_p(x,y,t) \tag{4}$$

$$\ddot{\mathbf{x}}_{stream}(x,y,t) = (u \cdot V_{p_x} + v \cdot V_{p_y})(x,y,t) \tag{5}$$

$$\kappa_{stream}(x,y,t) = \frac{\det[\dot{\mathbf{x}}_{stream}, \ddot{\mathbf{x}}_{stream}]}{\|\dot{\mathbf{x}}_{stream}\|^3}(x,y,t). \tag{6}$$

4 Path Lines

Path lines are obtained by setting out a particle and tracing its path in the unsteady flow. Therefore, path lines are projections of the tangent curves of V into a plane $t = \text{const}$. For every location and every time there is one and only one path line through it (except for critical points).

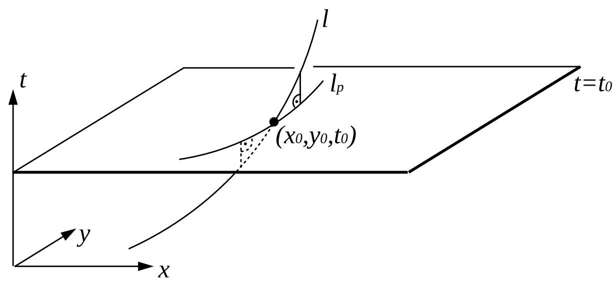

Fig. 3. Path line l_p of an unsteady flow

Consider figure 3. The curve l is the tangent curve of V through the point (x_0, y_0, t_0). The curve l_p is the projection of l into the plane $t = t_0$. The curvature of the path line through (x_0, y_0, t_0) is the curvature of l_p in this point. To compute it, we express the first and second derivative vectors of l in (x_0, y_0, t_0):

$$\dot{\mathbf{x}}_l = V \quad , \quad \ddot{\mathbf{x}}_l = u \cdot V_x + v \cdot V_y + a \cdot V_t. \tag{7}$$

Projecting $\dot{\mathbf{x}}_l$ and $\ddot{\mathbf{x}}_l$ into the plane $t = t_0$ and taking (2) into consideration we obtain the first and second derivatives of the path line in (x_0, y_0, t_0):

$$\dot{\mathbf{x}}_{path}(x, y, t) = V_p(x, y, t) \tag{8}$$

$$\ddot{\mathbf{x}}_{path}(x, y, t) = (u \cdot V_{p_x} + v \cdot V_{p_y} + V_{p_t})(x, y, t). \tag{9}$$

Then the curvature of the path line through (x_0, y_0, t_0) is

$$\kappa_{path}(x, y, t) = \frac{\det[\dot{\mathbf{x}}_{path}, \ddot{\mathbf{x}}_{path}]}{\|\dot{\mathbf{x}}_{path}\|^3}(x, y, t). \tag{10}$$

5 Streak Lines

A streak line is the location of all particles set out at one point at different times.

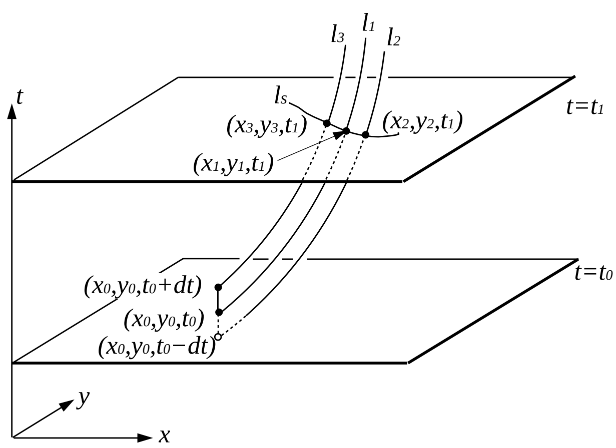

Fig. 4. Streak line l_s of an unsteady flow

Consider figure 4. Suppose a particle is set out at (x_0, y_0, t_0). The path of the particle is the tangent curve l_1 of the vector field V. l_1 might pass the location (x_1, y_1) at the time t_1 $(t_0 \leq t_1)$. We consider two more particles set out at (x_0, y_0) but a short time before and after t_0, i.e. we set out particles at $(x_0, y_0, t_0 - dt)$ and $(x_0, y_0, t_0 + dt)$. They follow the tangent curves l_2 and l_3 of V. Suppose l_2 passes the location (x_2, y_2) at the time $t = t_1$, and l_3 passes the location (x_3, y_3) at the time $t = t_1$. Then (x_1, y_1, t_1), (x_2, y_2, t_1) and (x_3, y_3, t_1) lie on a streak line l_s through (x_1, y_1, t_1). Converging dt to 0, we might compute tangent direction and curvature of l_s in (x_1, y_1, t_1).

Dealing with streak lines gives the following two problems:
a) A streak line through (x_1, y_1, t_1) is not uniquely defined. Another choice of t_0 might lead to another streak line through (x_1, y_1, t_1).
b) Computing a streak line through (x_1, y_1, t_1), we have to compute the tangent curve l_1 of V. This is only possible by integrating l_1 numerically - a procedure we wanted to avoid !
To avoid the problems a) and b) there are two solutions:
1) We consider only the special case $t_0 = t_1$. This way a streak line through (x_1, y_1, t_1) is uniquely defined and we do not have to trace the tangent curves. Unfortunately, in this case the streak line through (x_1, y_1, t_1) coincides with the stream line through (x_1, y_1, t_1) computed in section 3. So this case is of less interest.
2) Setting $t_0 = t_1$, the direction of the streak lines coincides with the direction of the stream lines: $\dot{\mathbf{x}}_{streak} = \dot{\mathbf{x}}_{stream} = V_p$. Setting $t_0 = t_1 - dt$, the direction of the streak lines might be $\dot{\mathbf{x}}_{dt}$, which usually differs from $\dot{\mathbf{x}}_{streak}$. Then we want to define the "curvature" of streak lines as a measure of how much the directions of $\dot{\mathbf{x}}_{streak}$ and $\dot{\mathbf{x}}_{dt}$ differ. In other words: The "curvature" of streak lines is a measure of how "strong" the directions of the streak lines change while varying the time t_0 of setting out the particles around t_1 of considering the streak lines. The choice of the concept "curvature" is justified in the following similarity to the usual curvature concept of curves: The curvature of a curve can be considered as a measure of how much the tangent direction changes while varying the location on the curve.

To compute the "curvature" of streak lines, we have to compute

$$\ddot{\mathbf{x}}_{streak} = \lim_{dt \to 0} \frac{\dot{\mathbf{x}}_{streak} - \dot{\mathbf{x}}_{dt}}{dt}. \tag{11}$$

From (11) we obtain

$$\dot{\mathbf{x}}_{streak}(x, y, t) = V_p(x, y, t) \quad , \quad \ddot{\mathbf{x}}_{streak}(x, y, t) = \begin{pmatrix} u_t(x, y, t) \\ v_t(x, y, t) \end{pmatrix} \tag{12}$$

and can compute the curvature of the streak lines by

$$\kappa_{streak}(x, y, t) = \frac{\det[\dot{\mathbf{x}}_{streak}, \ddot{\mathbf{x}}_{streak}]}{\|\dot{\mathbf{x}}_{streak}\|^3}(x, y, t). \tag{13}$$

6 Time Lines

Time lines are obtained by setting out particles located on a straight line at a fixed time and tracing them in the unsteady flow.

Consider figure 5. Suppose a particle is set out at (x_0, y_0, t_0). The path of the particle is the tangent curve l_1 of V. The curve l_1 might pass the location (x_1, y_1) at the time t_1 $(t_0 \leq t_1)$. We consider two more particles set out at the time $t = t_0$: $(x_0 - dx, y_0 - dy, t_0)$ and $(x_0 + dx, y_0 + dy, t_0)$. These points and (x_0, y_0, t_0) are located on a straight line in the plane $t = t_0$. Let these particles

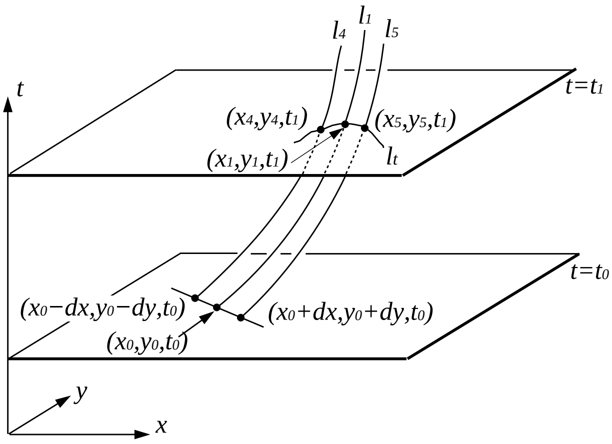

Fig. 5. Time line l_t of an unsteady flow

follow the tangent curves l_4 and l_5 of V. Suppose l_4 passes the location (x_4, y_4) at the time $t = t_1$, and l_5 passes the location (x_5, y_5) at the time $t = t_1$. Then (x_1, y_1, t_1), (x_4, y_4, t_1) and (x_5, y_5, t_1) lie on a time line l_t through (x_1, y_1, t_1).

The choice of a particular time line through (x_1, y_1, t_1) depends on two parameters: the choice of t_0 and the choice of the straight line in the plane $t = t_0$. Thus a time line through (x_1, y_1, t_1) is not uniquely defined. We therefore cannot compute its curvature as a local property.

7 The Visualization Technique

In sections 3-6 we have shown that for a given location (x_0, y_0) and a given time t_0 there is one and only one stream line and one and only one path line through it. Furthermore we were able to compute the curvature of the stream line and the path line in (x_0, y_0, t_0). With some more effort we were able to reduce the number of streak lines through (x_0, y_0, t_0) to one and could thus compute its curvature as a local property. In general, the curvatures of stream lines, path lines and streak lines in (x_0, y_0, t_0) differ.

For visualizing the flow we pick certain time steps $t_0, t_1, ...$ and compute and color code the curvature of stream lines, path lines and streak lines for every point of the domain. We use the color coding map described in [10] and section 1.

Figure 6 shows the flow of water in the bay area of the Baltic Sea near Greifswald, Germany (Greifswalder Bodden). The bay covers an area of 23×26 km. The maximal depth of the water is 12 m. The vectors of the sample points on a regular 115×103 grid are obtained by numerical simulation over 25 time steps. Therefore, the flow in this shallow water can be considered as an unsteady 2D flow.

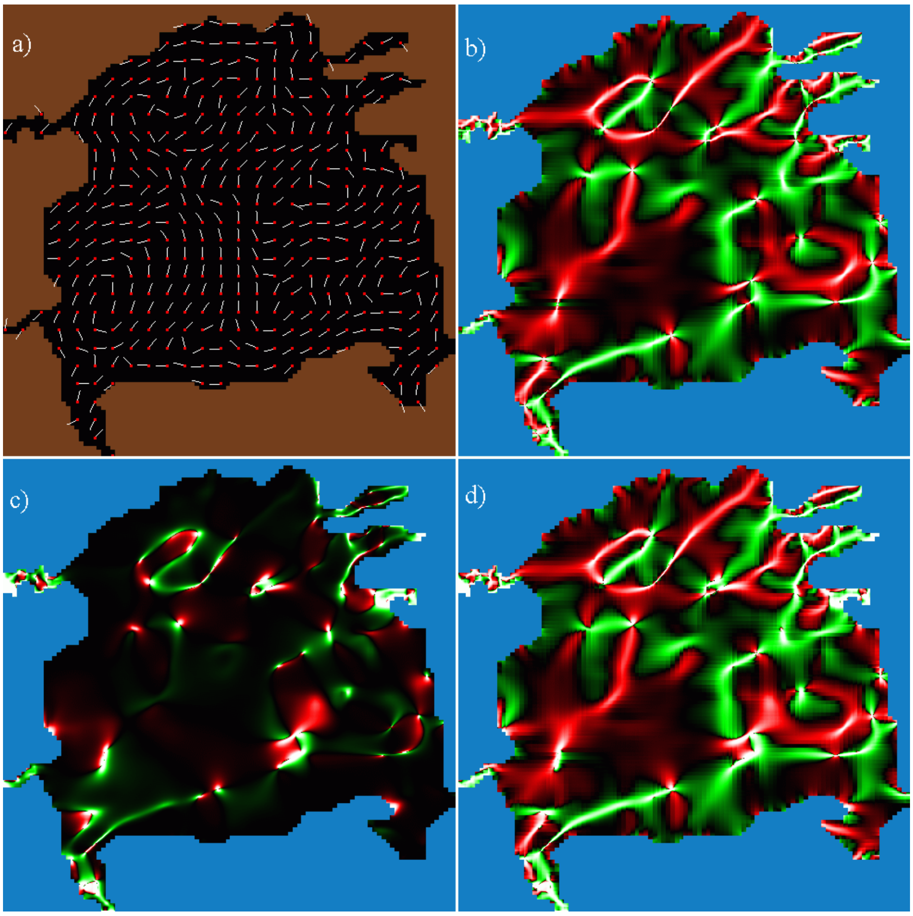

Fig. 6. Flow at time t_0. a) arrow plot; b) curvature of stream lines; c) curvature of streak lines; d) curvature of path lines

Figure 6a) shows an arrowplot of the vector field at a time of interest $t = t_0$. Figure 6b) shows the curvature of the stream lines for every point of the domain of the field at $t = t_0$. Figure 6c) shows the curvature of the streak lines, figure 6d) is the curvature of path lines. In all the curvature visualizations we can detect turbulent areas as brightly colored regions in the pictures. The critical points appear as highlights.

The curvatures of stream lines, path lines and streak lines show different aspects of the flow for a certain time of interest $t = t_0$. The stream line curvature shows how much the flow direction changes locally, i.e. the flow at a point and its environment is considered for a fixed time. The streak line curvature shows how much the flow direction changes temporarly at a fixed location. Here a bright color at a certain point of the vector field means that the flow direction is going to change rapidly in the next moments at this location. The path line curvature can be considered as a combination of stream line curvature and streak line curvature.

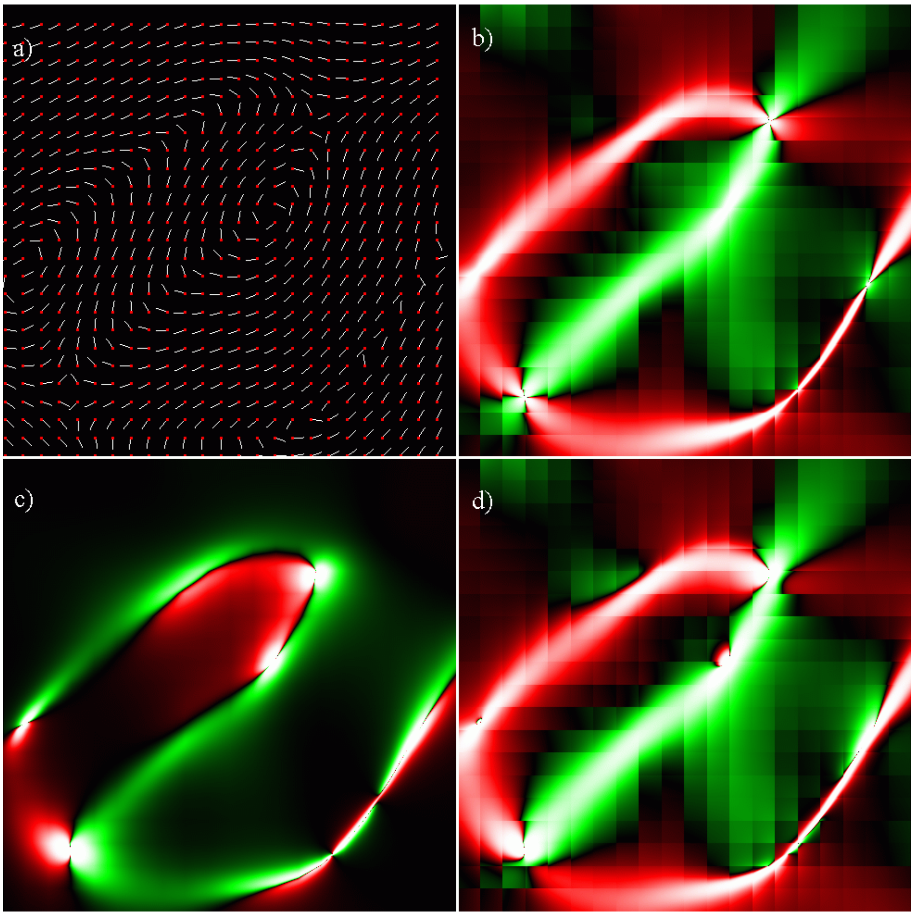

Fig. 7. Flow at time t_0 (magnification). a) arrow plot; b) curvature of stream lines; c) curvature of streak lines; d) curvature of path lines

Figure 7 shows the magnification of the turbulent area upper left in figure 6. Here the visualizations of the stream line curvature and path line curvature show clearly the underlying grid structure of the vector field. Due to the bilinear interpolation of the vector field, stream lines and path lines are not curvature continuous at the boundaries of the grid cells. In contrast to this, streak lines are curvature continuous in a bilinear vector field: the visualization of their curvature looks smooth (see figure 7c).

8 Conclusions

We have shown how to compute the curvatures of stream lines, path lines and streak lines of unsteady flow fields. We applied this as a visualization technique for flow data. The pictures obtained this way are without overloadings and ambiguities. They show clearly the turbulent regions of the flow as highlights. The local and temporal behavior of the flow can be deduced from the curvatures of stream lines and streak lines.

In future, the method should be applied to unsteady 3D flow data. Here we have to apply approaches of volume visualization for visualizing the curvature of the characteristic lines.

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